

ERGODICITY OF STOCHASTIC SHELL MODELS DRIVEN BY PURE JUMP NOISE

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ABSTRACT. In the present paper we study a stochastic evolution equation for shell (SABRA & GOY) models with pure jump Lévy noise $L = \sum_{k=1}^{\infty} l_k(t) e_k$ on a Hilbert space \mathbf{H} . Here $\{l_k; k \in \mathbb{N}\}$ is a family of independent and identically distributed (i.i.d.) real-valued pure jump Lévy processes and $\{e_k; k \in \mathbb{N}\}$ is an orthonormal basis of \mathbf{H} . We mainly prove that the stochastic system has a unique invariant measure. For this aim we show that if the Lévy measure of each component $l_k(t)$ of L satisfies a certain order and a non-degeneracy condition and is absolutely continuous with respect to the Lebesgue measure, then the Markov semigroup associated with the unique solution of the system has the strong Feller property. If, furthermore, each $l_k(t)$ satisfies a small deviation property, then 0 is accessible for the dynamics independently of the initial condition. Examples of noises satisfying our conditions are a family of i.i.d tempered Lévy noises $\{l_k; k \in \mathbb{N}\}$ and $\{l_k = W_k \circ G_k + G_k; k \in \mathbb{N}\}$ where $\{G_k; k \in \mathbb{N}\}$ (resp., $\{W_k; k \in \mathbb{N}\}$) is a sequence of i.i.d subordinator Gamma (resp., real-valued Wiener) processes with Lévy density $f_G(z) = (\vartheta z)^{-1} e^{-\frac{\vartheta}{z}} \mathbb{1}_{z>0}$. The proof of the strong Feller property relies on the truncation of the nonlinearity and the use of a gradient estimate for the Galerkin system of the truncated equation. The gradient estimate is a consequence of a Bismut-Elworthy-Li (BEL) type formula that we prove in the Appendix A of the paper.

1. INTRODUCTION

In many applied sciences such as aerodynamics, weather forecasting and hydrology, numerical investigation of three dimensional Navier-Stokes equations at high Reynolds' number is ubiquitous. Unfortunately, even with the most sophisticated scientific tools, it is a very challenging task to compute analytically or via direct numerical simulations the turbulent behavior of 3-D incompressible fluids. This is due to the large range of scale of motions that need to be resolved. To tackle this issue, several models of turbulence that can capture the physical phenomenon of turbulence in fluid flows at lower computability cost have been proposed over the last three decades. One class of these models of turbulence are the **shell models**. There are various kind of shell models, but the most popular in the physics and mathematics literature are the GOY and SABRA models. The shell models basically describe the evolution of complex Fourier-like components, denoted by u_n with the associated wave numbers denoted by k_n where the discrete index n is referred as the shell index, of a velocity field u . The evolution of the infinite sequence $\{u_n\}_{n=-1}^{\infty}$ is given by

$$(1) \quad \dot{u}_n(t) + \kappa k_n^2 u_n(t) + b_n(u(t), u(t)) = f_n(t, u(t)), \quad n = 1, 2, \dots$$

with $u_{-1} = u_0 = 0$ and $u_n(t) \in \mathbb{C}$ for $n \geq 1$. Here $\kappa \geq 0$ and in analogy with Navier-Stokes equations κ represents a kinematic viscosity; $k_n = k_0 \lambda^n$ ($\lambda > 1$) and f_n is a forcing term. The exact form of $b_n(u, v)$ varies from one model to the other. However in all the various models, it is assumed that $b_n(u, v)$ is chosen in such a way that

$$(2) \quad \Re \left(\sum_{n=1}^{\infty} b_n(u, v) \bar{v}_n \right) = 0,$$

where \Re denotes the real part and \bar{x} the complex conjugate of x .

In particular, we define the bilinear terms b_n as

$$b_n(u, v) = i(ak_{n+1}\bar{u}_{n+1}\bar{v}_{n+2} + bk_n\bar{u}_{n-1}\bar{v}_{n+1} - ak_{n-1}\bar{u}_{n-1}\bar{v}_{n-2} - bk_{n-1}\bar{u}_{n-2}\bar{v}_{n-1})$$

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in the GOY model (see [24, 38]) and by

$$b_n(u, v) = -i(ak_{n+1}\bar{u}_{n+1}v_{n+2} + bk_n\bar{u}_{n-1}v_{n+1} + ak_{n-1}u_{n-1}v_{n-2} + bk_{n-1}u_{n-2}v_{n-1})$$

in the SABRA model (see [35]). Here a, b are real numbers. Note that equation (2) implies a formal law of conservation of energy in the inviscid and unforced form of (1). The shell models have similar properties to 2D fluids. In fact, they basically consist of infinitely many differential equations having a structure similar to the Fourier representation of the Navier-Stokes equations. They are constructed in such a way that they and the Navier-Stokes equations at high Reynolds' number exhibit similar statistical properties. Indeed both shell models and Navier-Stokes models have a finite number of degrees of freedom, see, for instance, [15]. Another feature that shell models share with the Navier Stokes equations is the so called *determining modes*, see the pioneering work of Foias and Prodi [23] for the case of Navier-Stokes equations and [15] for the shell models. Furthermore, the interactions in the Fourier space for the shell models are local and therefore are easier to handle. As such shell models are much simpler than the Navier-Stokes equations and are more suitable for the analytical and numerical investigation towards the understanding of turbulence. Due to these facts these models and their stochastic counterparts have been the subject of intensive numerical and analytical studies during the last two decades. We refer, for instance, to the works of Barbato et al [4], Bessaih et al [8], Constantin et al [15], and Ditlevsen [17] for more recent and detailed review of results related to the physical and mathematical theory of shell models.

In recent years the mathematical analysis of stochastic partial differential equations driven by Lévy processes began to draw more and more attention. There are several examples where the Gaussian noise is not well suited to represent realistically external forces. For example, if the ratio between the time scale of the deterministic part and that of the stochastic noise is large, then the temporal structure of the forcing in the course of each event has no influence on the overall dynamics, and - at the time scale of the deterministic process - the external forcing can be modeled as a sequence of episodic instantaneous impulses. This happens for example in Climatology (see, for instance, [30]). Often the noise observed by time series is typically asymmetric, heavy-tailed and has non trivial kurtosis. These are all features which cannot be captured by a Gaussian noise, but by a Lévy noise with appropriate parameters. From the mathematical point of view, Lévy randomness requires other techniques, and is intricate and far from amenable to mathematical analysis. Despite these facts the mathematical study of the long-time behavior, in particular ergodicity, of SPDEs with Lévy noise are still at its infancy. This is mainly due to the fact that the numerous results for Wiener driven models cannot be in general transferred to SPDEs driven by Lévy noise. The analysis of the long-time behavior of SPDEs is more complicated for Lévy driven SPDEs. In addition, the dynamical behavior of SPDEs changes essentially, if the Brownian noise is replaced by Lévy noise. E.g. Imkeller and Pavlyukevich investigated in [31, 30] the dynamical behavior of systems driven by a Lévy noise and showed that the escape times from certain potentials are exponentially distributed and differ essentially from the escape times of the corresponding dynamical systems driven by Brownian noise. One should note that there are now several papers treating the ergodicity of nonlinear SPDEs with Lévy noise, see for example [12], [33], [37], [40], [42], [43] and [41].

In the present paper we investigate the ergodicity of stochastic shell models driven by random external forcing of jump type. More precisely, we are interested in a model equation of the form

$$(3) \quad \begin{aligned} d\mathbf{u}(t) + [\kappa A\mathbf{u}(t) + \mathbf{B}(\mathbf{u}(t), \mathbf{u}(t))]dt &= \sum_{k=1}^{\infty} \left(\int_{\mathbb{R}_0} z d\tilde{\eta}_k(z, t) \right) \beta_k e_k \\ \mathbf{u}(0) &= \xi \in \mathbf{H}, \end{aligned}$$

where κ is a positive number, $\{e_k, k = 1, 2, \dots\}$ is the orthonormal basis of a given Hilbert space \mathbf{H} and

$$\int_{\mathbb{R}_0} z d\tilde{\eta}_k(z, t) := \int_{\mathbb{R}_0} \mathbb{1}_{\{|z| \leq 1\}} z \tilde{\eta}_k(dz, dt) + \int_{\mathbb{R}_0} \mathbb{1}_{\{|z| > 1\}} z \eta_k(dz, dt).$$

In (3), A is a linear map and \mathbf{B} is a bilinear map on the underlying Hilbert space \mathbf{H} . The family $\{\eta_k; k = 1, 2, \dots\}$ represents a family of mutually independent Poisson random measures with σ -finite Lévy measures $\{\nu_k; k = 1, 2, \dots\}$ on $\mathbb{R}_0 := \mathbb{R} \setminus \{0\}$. For each k the symbol $\tilde{\eta}_k$ represents the compensated Poisson random

measure associated to η_k and the family of compensators is denoted by $\{\nu_k(dz)dt; k = 1, 2, \dots\}$. The family $\{\beta_k; k = 1, 2, \dots\}$ is a family of positive numbers representing the roughness of the noise. The maps \mathbf{A} and \mathbf{B} are carefully chosen so that it can model the nonlinear terms of the GOY and SABRA shell models defined previously.

In this paper, we mainly prove that if the Lévy measure of each component $l_k(t)$ of L satisfies a certain order and non-degeneracy condition, and is absolutely continuous w.r.t. the Lebesgue measure and if each Lévy process $l_k(t) := \int_{\mathbb{R}_0} z d\bar{\eta}_k(z, t)$ satisfies a small deviation property, then the stochastic evolution equations (3) has a unique invariant measure (see Theorem 3.3). Examples of noises satisfying our conditions are a family of i.i.d tempered Lévy noises $\{l_k; k \in \mathbb{N}\}$ and $\{l_k = W_k \circ G_k + G_k; k \in \mathbb{N}\}$ where $\{G_k; k \in \mathbb{N}\}$ (resp., $\{W_k; k \in \mathbb{N}\}$) is a sequence of i.i.d subordinator Gamma (resp., real-valued Wiener) processes with Lévy density $f_G(z) = (\vartheta z)^{-1} e^{-\frac{z}{\vartheta}} \mathbb{1}_{z>0}$. We mainly show that the Markovian semigroup associated with the solution of (3) has strong Feller property and that $0 \in \mathbf{H}$ is an accessible point for the dynamic. The strong Feller property is the most challenging part of the proof. The strategy of the proof of this result is based on the work [22]. Namely, we firstly truncate the nonlinearity (3) and show that the Galerkin approximation of this modified/truncated version of (3) has the strong Feller property. This was achieved thanks to a Bismut-Elworthy-Li (BEL) type formula (see Lemma A.3) that we state and prove in Appendix A. Lemma A.3 is very similar to [45, Theorem 1]. However, for each $n \in \mathbb{N}$ our Galerkin equations is a system of stochastic differential equations driven by random measures with Lévy measure on \mathbb{R}^n which, in contrast to [45], do not necessarily have a smooth density. Nevertheless, we should note that the idea of the proof of Lemma A.3 is based on some modifications of [45, Proof of Theorem 1] and some arguments (change of measures) from [32]. The main assumptions for this BEL type formula to hold is an order condition type, a non-degeneracy (see Assumption 2.3-(iv)) and absolute continuity w.r.t. the Lebesgue measure (Assumption 2.3-(ii)) of the Lévy measure of each $l_k(t)$. Secondly, we prove that the truncated equations itself has the strong Feller property. This result is based on càdlàg property of stochastic convolution $\mathfrak{S}(t)$ which is solution to the following equation

$$d\mathfrak{S}(t) + \kappa \mathbf{A} \mathfrak{S}(t) dt = \sum_{k=1}^{\infty} \left(\int_{\mathbb{R}_0} z d\bar{\eta}_k(z, t) \right) \beta_k e_k, \quad \mathfrak{S}(0) = 0.$$

Here recent results about time regularity of stochastic convolutions proved in [39] play an important role. Thirdly, since the solution to the original equation has good moment estimates we can show that the strong Feller property is preserved when we remove the truncation function. To complete the proof of our main result we show in Proposition 3.8 that any ball centered at 0 with sufficiently large radius is visited, with positive probability, by the process \mathbf{u} independently of the initial condition $\xi \in \mathbf{H}$. This fact holds under the condition that each one dimensional Lévy process $l_k(t)$ has the small deviation property (see Proposition 2.6).

We should note that the nonlinear term of (3) does not fall in the framework of the papers [12, 37, 42, 43, 41]. The paper [19] and the book [34] studied the uniqueness of invariant measure associated to the Markov semigroup of the stochastic Navier-Stokes equations with Lévy noise. In [19] the driving noise is the sum of a non-degenerate Wiener noise and an infinite activity jump process. Thanks to the non-degeneracy of the Wiener noise the gradient estimate method in [22, 21] can be adapted to their framework. In our case we closely follow the scheme in [22] for the proof of the smoothing property of the semigroup, but we have to prove a BEL-type formula for pure jump noise. The authors of [34] state that stochastic Navier-Stokes equations with Poisson process as a noise term has a unique invariant measure. They also provide the rate of convergence to the invariant measure. The proof of these results follows from the arguments in [37] which are very sophisticated and complicated to be explained here. Since the Lévy measure of a Poisson process is a finite measure and we consider Lévy processes with σ -finite Lévy measure, the proof of [37] could not be used for our model.

To close this introduction we give the structure of this paper. In Section 2 we define most of the notations used in this paper and the assumptions frequently imposed throughout the paper. The main result (see Theorem 3.10) is stated and proved in Section 3. The proof of this main theorem relies on two important propositions (Proposition 3.6 and Proposition 3.8) that are also stated and proved in Section 3. Section 4 is

devoted to the analytical study of the truncated version of (13) (see Eq. (23)). There, we mainly prove that the finite dimensional approximation of the truncated equations satisfy the strong Feller property which is preserved by passage to the limit. In Appendix A, we prove a Bismut-Elworthy-Li type formula for system of SDEs driven by pure jump noise. In Appendix B we prove an estimate for the gradient of the semigroup of the system of SDEs from Appendix A. In Appendix C we derive the necessary convergence which enables us to transfer the strong Feller property from the semigroup of the Galerkin approximation of the truncated equations to the semigroup of the infinite dimensional truncated equation.

2. NOTATION AND ASSUMPTIONS

In this section we will introduce the necessary notation and assumptions in this paper. We will mainly follow the notation in [6].

Throughout this work we will identify the field of complex numbers \mathbb{C} with \mathbb{R}^2 . That is, any complex number of the form $x = x_1 + ix_2$ will be identified with $(x_1, x_2) \in \mathbb{R}^2$. As usual, for $x = (x_1, x_2) \in \mathbb{R}^2$ we set $|x|^2 = x_1^2 + x_2^2$ and $x \cdot y = x_1 y_1 + x_2 y_2$ is the scalar product in \mathbb{R}^2 . For a Banach space B we denote by B^* its dual space.

Let \mathbf{H} be the space defined by

$$\mathbf{H} = \{\mathbf{u} = (\mathbf{u}_1, \mathbf{u}_2, \dots) \in (\mathbb{R}^2)^\infty : \sum_{n=1}^{\infty} |\mathbf{u}_n|^2 < \infty\}.$$

This is a Hilbert space and we denote its norm by $|\cdot|$ and its scalar product by $\langle \mathbf{u}, \mathbf{v} \rangle = \sum_{n=1}^{\infty} \mathbf{u}_n \cdot \mathbf{v}_n$.

Let A be a linear map with domain $D(A)$ on \mathbf{H} . We impose the following set of conditions on A .

Assumptions 2.1. (i) We assume that A is a self-adjoint positive operator and its domain $D(A)$ is dense and compact in \mathbf{H} .

(ii) We assume that there exists k_0 and $\lambda > 1$ such that the eigenvalues of A are given by

$$\lambda_j = k_0 \lambda^{2j}.$$

(iii) We also suppose that the eigenfunctions $\{e_1, e_2, \dots\}$ of A form an orthonormal basis of \mathbf{H} .

With Assumption 2.1-(i) the fractional power operators A^γ , $\gamma \geq 0$ are well-defined; they are also self-adjoint, positive and invertible with inverse $A^{-\gamma}$. We denote by $\mathbf{V}_\gamma := D(A^{\frac{\gamma}{2}})$, $\gamma \geq 0$ the domain of A^γ . It is a Hilbert space endowed with the graph norm. The dual space \mathbf{V}_γ^* of \mathbf{V}_γ , $\gamma \geq 0$, w.r.t. to the inner product of \mathbf{H} can be identified with $D(A^{-\frac{\gamma}{2}})$. For $\gamma = \frac{1}{2}$ we set $\mathbf{V} := \mathbf{V}_1$ and we denote its norm by $\|\cdot\| := |\cdot| + |A^{\frac{1}{2}} \cdot|$. Observe also that Assumption 2.1 implies the following Poincaré type inequality

$$(4) \quad |\cdot|^2 \leq \lambda_1^{-1} \|\cdot\|^2,$$

where λ_1 is the smallest eigenvalue of A . Thus the norm $\|\cdot\|$ is equivalent to $|A^{\frac{1}{2}} \cdot|$.

When identifying \mathbf{H} with its dual \mathbf{H}^* we have the Gelfand triple $\mathbf{V} \subset \mathbf{H} \subset \mathbf{V}^*$. We denote by $\langle \mathbf{u}, \mathbf{v} \rangle$ the duality between \mathbf{V}^* and \mathbf{V} such that $\langle \mathbf{u}, \mathbf{v} \rangle = (\mathbf{u}, \mathbf{v})$ for $\mathbf{u} \in \mathbf{H}$ and $\mathbf{v} \in \mathbf{V}$.

Now, let $\mathbf{B} : \mathbf{H} \times \mathbf{H} \rightarrow \mathbf{V}^*$ be a bilinear map satisfying the following set of conditions.

Assumptions 2.2. We assume that $\mathbf{B} : \mathbf{H} \times \mathbf{H} \rightarrow \mathbf{V}^*$ is a bilinear map satisfying the following three properties.

(a) There exists a number $C_0 > 0$ such that for any $\mathbf{u}, \mathbf{v} \in \mathbf{H}$

$$(5) \quad \|\mathbf{B}(\mathbf{u}, \mathbf{v})\|_{\mathbf{V}^*} \leq C_0 |\mathbf{u}| |\mathbf{v}|.$$

(b) There exists a constant $C_1 > 0$ such that

$$(6) \quad |\mathbf{B}(\mathbf{u}, \mathbf{v})| \leq \begin{cases} C_1 \|\mathbf{u}\| |\mathbf{v}| & \text{for any } \mathbf{u} \in \mathbf{V}, \mathbf{v} \in \mathbf{H}, \\ C_1 |\mathbf{u}| \|\mathbf{v}\| & \text{for any } \mathbf{v} \in \mathbf{V}, \mathbf{u} \in \mathbf{H}. \end{cases}$$

(c) For any $\mathbf{u} \in \mathbf{H}, \mathbf{v} \in \mathbf{V}$

$$(7) \quad \langle \mathbf{B}(\mathbf{u}, \mathbf{v}), \mathbf{v} \rangle = 0.$$

In our framework \mathbf{B} is defined by $\mathbf{B} : (\mathbb{R}^2)^\infty \times (\mathbb{R}^2)^\infty \rightarrow (\mathbb{R}^2)^\infty$

$$\mathbf{B}(u, v) = (B_1(u, v), B_2(u, v), \dots),$$

where $B_n = (B_{n,1}, B_{n,2})$ and $B_{n,1}$ and $B_{n,2}$ are, respectively, the real parts and the imaginary parts of the b_n given in the previous section. For instance, in the SABRA model

$$(8) \quad \begin{aligned} B_{1,1}(u, v) &= ak_2[-u_{2,2}v_{3,1} + u_{2,1}v_{3,2}] \\ B_{1,2}(u, v) &= -ak_2u_2 \cdot v_3 \end{aligned}$$

$$(9) \quad \begin{aligned} B_{2,1}(u, v) &= ak_3[-u_{3,2}v_{4,1} + u_{3,1}v_{4,2}] + bk_2[-u_{1,2}v_{3,1} + u_{1,1}v_{3,2}] \\ B_{2,2}(u, v) &= -ak_3u_3 \cdot v_4 - bk_2u_1 \cdot v_3 \end{aligned}$$

and for $n > 2$

$$(10) \quad \begin{aligned} B_{n,1}(u, v) &= ak_{n+1}[-u_{n+1,2}v_{n+2,1} + u_{n+1,1}v_{n+2,2}] \\ &\quad + bk_n[-u_{n-1,2}v_{n+1,1} + u_{n-1,1}v_{n+1,2}] \\ &\quad + ak_{n-1}[u_{n-1,2}v_{n-2,1} + u_{n-1,1}v_{n-2,2}] \\ &\quad + bk_{n-1}[u_{n-2,2}v_{n-1,1} + u_{n-2,1}v_{n-1,2}], \end{aligned}$$

$$(11) \quad \begin{aligned} B_{n,2}(u, v) &= -ak_{n+1}[u_{n+1,1}v_{n+2,1} + u_{n+1,2}v_{n+2,2}] \\ &\quad - bk_n[u_{n-1,1}v_{n+1,1} + u_{n-1,2}v_{n+1,2}] \\ &\quad - ak_{n-1}[u_{n-1,1}v_{n-2,1} - u_{n-1,2}v_{n-2,2}] \\ &\quad - bk_{n-1}[u_{n-2,1}v_{n-1,1} - u_{n-2,2}v_{n-1,2}]. \end{aligned}$$

It is proved in [6] that the maps \mathbf{B} for GOY and SABRA shell models defined as above satisfy Assumption 2.2.

Let $\mathfrak{P} = (\Omega, \mathcal{F}, \mathbb{P}, \mathbb{F})$ be a filtered complete probability space such that the filtration $\mathbb{F} = (\mathcal{F}_t)_{t \geq 0}$ satisfies the usual condition.

Let $\eta := \{\eta_1, \eta_2, \dots\}$ be a family of mutually independent Poisson random measures defined on \mathfrak{P} with Lévy measures $\{\nu_1, \nu_2, \dots\}$. We assume that each ν_j is a σ -finite measure on $\mathbb{R}_0 := \mathbb{R} \setminus \{0\}$. We denote by $\{\nu_1(dz_1)dt, \nu_2(dz)dt, \dots\}$ the family of compensators of the elements of η and $\{\tilde{\eta}_1, \tilde{\eta}_2, \dots\}$ the family of compensated Poisson random measures associated to the elements of η . To shorten notation we will use the following notations $d\eta_j(z, t) := \eta_j(dz, dt)$, $d\tilde{\eta}_j(z, t) := \tilde{\eta}_j(dz, dt)$ and $d\nu_j(z)dt := \nu_j(dz)dt$ for any $j \in \{1, 2, \dots\}$. We will also use the notation

$$d\tilde{\eta}_j(z, t) = \mathbf{1}_{|z_j| \leq 1} d\tilde{\eta}_j(z, t) + \mathbf{1}_{|z_j| > 1} d\eta_j(z, t),$$

for any $j \in \{1, 2, \dots\}$.

Assumptions 2.3. (i) The Poisson random measures $\eta_j, j \in \{1, 2, \dots\}$ are independent and identically distributed. This means in particular that there exists a Lévy measure ν such that $\nu_j(dz_j) = \nu(dz)$ for any $j \in \{1, 2, \dots\}$.

(ii) There exists a C^1 function $g : \mathbb{R} \rightarrow [0, \infty)$ and a constant $C > 0$ such that $\nu(dz) = g(z)dz$ and for any $p \geq 1$ we have

$$\left| \frac{g'(z)}{g(z)} \right|^p \leq C(1 + |z|^{-p}), \quad z \in \mathbb{R}_0.$$

(iii) As $|z| \rightarrow \infty$ we have $z^2 g(z) \rightarrow 0$. Also

$$\int_{\mathbb{R}_0} \left[|z|^q \mathbf{1}_{(|z| \leq 1)} + |z|^q \mathbf{1}_{(|z| > 1)} \right] \nu(dz) < \infty,$$

for any $q \geq 1$.

(iv) Furthermore, there exists a constant $\alpha > 0$ such that for any $y \in \mathbb{R}$

$$\liminf_{\varepsilon \searrow 0} \varepsilon^\alpha \int_{\mathbb{R}_0} (|z \cdot y / \varepsilon|^2 \wedge 1) \nu(dz) > 0.$$

Remark 2.4. Assumption 2.3(iii)-Assumption 2.3(iv) are very similar to [45, Assumption 1] and ensure the validity of a Bismut-Elworthy-Li formula that we will prove in Appendix A. Assumption 2.3(iv) is called in some literature the order and non-degeneracy condition (see [45, Remark 2.2]).

The following concept plays an essential role in the proof of our main result. The following definitions is taken from [44].

Definitions 2.5. (1) A real-valued Lévy process $\{l(t); t \geq 0\}$ has the small deviation property if for any $T > 0$ and $\varepsilon > 0$,

$$\mathbb{P}\left(\sup_{t \in [0, T]} |l(t)| < \varepsilon\right) > 0.$$

(2) A Lévy measure ρ on \mathbb{R}_0 is said of *type (I)* if

$$\int_{-1}^1 |z| \rho(dz) < \infty.$$

(3) A real-valued pure jump Lévy process is a Lévy process without continuous part.

We recall the characterization of the small deviation property for real-valued pure jump Lévy processes in the following proposition (see [44, Théorème, pp 157], [3, Proposition 1.1]).

Proposition 2.6. *A real-valued pure jump Lévy process $\{l(t); t \geq 0\}$ admits the small deviation property if its Lévy measure ρ is not of type (I) or it is of type (I) and, for $\mathcal{E} = -\int_{|z| \leq 1} z \rho(dz)$, we have*

- $\mathcal{E} = 0$, or
- $\mathcal{E} > 0$ and $\rho(-\varepsilon \leq z < 0) \neq 0$ for all $\varepsilon > 0$, or
- $\mathcal{E} < 0$ and $\rho(0 < z \leq \varepsilon) \neq 0$ for all $\varepsilon > 0$.

Definition 2.7. If a Lévy measure ρ on \mathbb{R}_0 satisfies the characterizations given in Proposition 2.6, then we will say that it satisfies the small deviation property condition.

Now we introduce an additional assumption for the Lévy measure ν given in Assumption 2.3.

Assumptions 2.8. The Lévy measure ν satisfies the small deviation property condition (see Proposition 2.6).

Before we state the final assumption for the paper we give some basic examples that satisfy Assumption 2.3 and Assumption 2.8.

Remark 2.9. (1) Let $c_+, C_-, \beta_+, \beta_- > 0$ and $\alpha \in [0, 1)$. Define the general tempered Lévy measure

$$\nu(dz) = c_+ |z|^{-1-\alpha} e^{-\beta_+ |z|} \mathbb{1}_{z>0} + c_- |z|^{-1} e^{-\beta_- |z|} \mathbb{1}_{z<0}.$$

The simplest choice for the parameters $c_+, C_-, \beta_+, \beta_- > 0$ for ν to satisfy Assumption 2.3 and Assumption 2.8 is

$$c_+ = c_-, \beta_+ = \beta_-, \text{ and } \alpha \in [0, 1).$$

This choice corresponds to a symmetric tempered stable Lévy measure. This claim can be checked by elementary arguments. With the help of a good software (Mathematica for instance) one can also play with $c_+, C_-, \beta_+, \beta_- > 0$ and give other choices which are more complicated than the one above. We leave this as an exercise for the interested reader.

(2) The components of the noise in (13) can be replaced with the following ones

$$(12) \quad \ell_k(t) = \sigma W_k(G_k(t)) + \theta G_k(t), \quad \sigma > 0, \theta \in \mathbb{R}, t \in [0, \infty), k \in \mathbb{N},$$

where $\{W_k; k \in \mathbb{N}\}$ is a family of i.i.d standard Brownian motions and $\{G_k; k \in \mathbb{N}\}$ is a family of i.i.d Gamma processes with Lévy measure $\nu_G(dz) = (\vartheta z)^{-1} e^{-\frac{z}{\vartheta}} \mathbb{1}_{z>0} dz$, $\vartheta > 0$. In fact it was shown in [29, Chapter 10] that each ℓ_k is a pure jump Lévy noise which is identical in law to a variance Gamma process $\tilde{\ell}_k$ having a Lévy measure

$$\nu(dz) = \left(c |z|^{-1} e^{-\beta_+ |z|} \mathbb{1}_{z>0} + c |z|^{-1} e^{-\beta_- |z|} \mathbb{1}_{z<0} \right) dz,$$

with $c = \frac{1}{\vartheta}$, $\beta_+ = 2c/(\sqrt{2\sigma^2/\vartheta + \theta^2} + \theta)$, $\beta_- = 2c/(\sqrt{2\sigma^2/\vartheta + \theta^2} - \theta)$. If we take $\theta = 0$, then we are in the situation of symmetric tempered stable process with $\alpha = 0$. We can also play with the parameters ϑ , σ and θ to give other examples, but we again leave it for the interested reader.

The final assumption on our model is the following.

Assumptions 2.10.

Let $\{\beta_j; j = 1, 2, \dots\}$ be a family of positive numbers such that there exists $\theta \in (\frac{1}{4}, \frac{1}{2})$ and

$$\beta_j = \lambda_j^{-\theta}.$$

To close this section we also introduce the following additional notations. For a Banach space B we denote respectively by $B_b(B)$, $C_b(B)$, and $C_b^2(B)$ the space of bounded and measurable functions, the space of continuous and bounded functions, and the space of bounded and twice Fréchet differentiable functions on B and taking values in \mathbb{R} . For two Banach spaces B_1 and B_2 we denote by $C_b^2(B_1, B_2)$ the space of bounded and twice Fréchet differentiable functions on B_1 and taking values in B_2 . Throughout this paper, $B_B(x, r)$ is the ball of radius r centered at $x \in B$; when $x = 0$ we simply write $B_B(r)$.

3. ERGODICITY OF THE STOCHASTIC SHELL MODELS

The aim of this paper is to study the uniqueness of the invariant measure associated to the solution of the abstract evolution equation given by

$$(13) \quad \begin{aligned} d\mathbf{u}(t) + [\kappa \mathbf{A}\mathbf{u}(t) + \mathbf{B}(\mathbf{u}(t), \mathbf{u}(t))]dt &= \sum_{k=1}^{\infty} \int_{\mathbb{R}_0} \beta_k z e_k d\bar{\eta}_k(z, t) \\ \mathbf{u}(0) &= \xi, \end{aligned}$$

where κ is a positive constant and the Poisson random measures η_j , $j \in \{1, 2, \dots\}$ are as above. In what follows we set

$$L(t) = \sum_{k=1}^{\infty} \int_0^t \int_{\mathbb{R}_0} \beta_k z e_k d\bar{\eta}_k(z, s).$$

We first introduce the notion of solution and give the conditions under which a solution \mathbf{u} to Eq. (13) exists.

Definition 3.1. Let $T > 0$ be an arbitrary real number. An \mathbb{F} -adapted process \mathbf{u} is called a solution of Eq. (13) if the following conditions are satisfied:

- (i) $\mathbf{u} \in L^2(0, T; \mathbf{H})$ \mathbb{P} -almost surely,
- (ii) the following equality holds for every $t \in [0, T]$ and \mathbb{P} -a.s.,

$$(14) \quad (\mathbf{u}(t), \phi) = (\xi, \phi) - \int_0^t ((\kappa \mathbf{A}\mathbf{u}(s) + \mathbf{B}(\mathbf{u}(s), \mathbf{u}(s))), \phi) ds + \langle L(t), \phi \rangle,$$

for any $\phi \in \mathbf{V}$.

Remark 3.2. Owing to Assumption 2.2-(a) the nonlinear term $\int_0^t (\mathbf{B}(\mathbf{u}(s), \mathbf{u}(s)), \phi) ds$ makes sense whenever $\phi \in \mathbf{V}$ and $\mathbf{u} \in L^2(0, T; \mathbf{H})$. Also, thanks to Assumption 2.10 and [40, Theorem 4.40] it can be easily checked that the Levy process L lives in $D(A^{-\frac{1}{2}})$. Thus $\langle L(t), \phi \rangle$ makes sense for any $\phi \in \mathbf{V}$.

3.1. Resolvability of problem (13). We state and prove the following proposition.

Proposition 3.3. *In addition to Assumption 2.1, Assumption 2.2 and Assumption 2.10 we assume that the items (i) and (iii) of Assumption 2.3 hold. Then, problem (13) has a unique solution \mathbf{u} which has a càdlàg modification in \mathbf{H} . For any $t \geq 0$, $\xi \in \mathbf{H}$ and $p \in \{2, 4\}$ there exists a constant $C := C(t, \xi) > 0$ such that*

$$(15) \quad \mathbb{E} \sup_{s \in [0, t]} |\mathbf{u}(s, \xi)|^p + \kappa \mathbb{E} \int_0^t |\mathbf{u}(s, \xi)|^{p-2} |A^{\frac{1}{2}} \mathbf{u}(s, \xi)|^2 ds \leq C.$$

Moreover, \mathbf{u} is a Markov process having the Feller property.

Proof. First we will prove that (15) holds for the Galerkin approximation of (13).

For each $n \in \mathbb{N}$ let

$$\mathbf{H}_n := \text{Linspan}\{e_1, \dots, e_n\},$$

and $\Pi_n : \mathbf{V}^* \rightarrow \mathbf{H}_n$ be the orthogonal projection defined by

$$\Pi_n \mathbf{v} := \sum_{k=1}^n \langle \mathbf{v}, e_k \rangle e_k, \text{ for any } \mathbf{v} \in \mathbf{V}^*.$$

Throughout this paper, we will identify \mathbf{H}_n with \mathbb{R}^n .

Owing to [1, Theorem 3.1], for each $n \in \mathbb{N}$ there exists a càdlàg process \mathbf{u}_n which solves the system of stochastic differential equations

$$d\mathbf{u}_n + [\kappa A \mathbf{u}_n(t) + \Pi_n \mathbf{B}(\mathbf{u}_n(t), \mathbf{u}_n(t))]dt = \sum_{k=1}^n \beta_k dl_k(t) e_k, \quad \mathbf{u}_n(0) = \Pi_n \xi,$$

where

$$dl_k(t) = \int_{|z|<1} z \tilde{\eta}_k(dz, dt) + \int_{|z|\geq 1} z \eta_k(dz, dt) =: \int_{\mathbb{R}_0} z d\tilde{\eta}_k(z, t).$$

Applying Itô's formula to $|\mathbf{u}_n(t)|^p$ and using Assumption 2.2-(c) yield

$$d|\mathbf{u}_n(t)|^p + p\kappa |A^{\frac{1}{2}} \mathbf{u}_n(t)|^2 |\mathbf{u}_n(t)|^{p-2} dt = dI_1(t) + dI_2(t) + I_3(t)dt,$$

where

$$\begin{aligned} dI_1(t) &:= \sum_{k=1}^n \int_{|z|<1} (|\mathbf{u}_n(t) + \beta_k z e_k|^p - |\mathbf{u}_n(t)|^p) d\tilde{\eta}_k(z, t), \\ dI_2(t) &:= \sum_{k=1}^n \int_{|z|\geq 1} (|\mathbf{u}_n(t) + \beta_k z e_k|^p - |\mathbf{u}_n(t)|^p) d\eta_k(z, t), \\ I_3(t) &:= \sum_{k=1}^n \int_{|z|<1} (|\mathbf{u}_n(t) + \beta_k z e_k|^p - |\mathbf{u}_n(t)|^p - \Psi_{\mathbf{u}_n, t}[\beta_k z e_k]) \nu(dz) dt, \end{aligned}$$

and $\Psi_{\mathbf{u}_n, t}[h] = p|\mathbf{u}_n(t)|^{p-2} \langle \mathbf{u}_n(t), h \rangle$ for any $h \in \mathbf{H}$ and $t \geq 0$. Since $\eta_k(dz, dt)$, $k = 1, 2, \dots$ are non-negative measures and

$$\begin{aligned} |\mathbf{u}_n(t) + \beta_k z e_k|^p - |\mathbf{u}_n(t)|^p &\leq \beta_k^p |z|^p + C_p \sum_{r=1}^{p-1} \beta_k^r |z|^r |\mathbf{u}_n(t)|^{p-r} \\ &\leq \beta_k^p |z|^p + C_p \sum_{r=1}^{p-1} \beta_k^r |z|^r (1 + |\mathbf{u}_n(t)|^p) \end{aligned}$$

we have

$$|I_2(t)| \leq C_p \sum_{k=1}^n \left(\int_0^t \int_{|z|\geq 1} \left(\sum_{r=1}^p \beta_k^r |z|^r \right) d\eta_k(z, s) + \int_0^t \int_{|z|\geq 1} \left(C_p \sum_{r=1}^{p-1} \beta_k^r |z|^r |\mathbf{u}_n(s)|^p \right) d\eta_k(z, s) \right).$$

Hence,

$$\begin{aligned} \mathbb{E} \sup_{s \in [0, t]} |I_2(s)| &\leq C_p \sum_{k=1}^n \mathbb{E} \left(\int_0^t \int_{|z|\geq 1} \left(\sum_{r=1}^p \beta_k^r |z|^r \right) d\eta_k(z, s) \right. \\ &\quad \left. + \int_0^t \int_{|z|\geq 1} \left(C_p \sum_{r=1}^{p-1} \beta_k^r |z|^r |\mathbf{u}_n(s)|^p \right) d\eta_k(z, s) \right) \\ &\leq t C_p C_\nu \sum_{r=1}^p \left[\sum_{k=1}^\infty \beta_k^r \right] + C_p C_\nu \sum_{r=1}^p \left[\sum_{k=1}^\infty \beta_k^r \right] \times \mathbb{E} \int_0^t |\mathbf{u}_n(s)|^p ds, \end{aligned}$$

where for any $q \geq 1$ we have set $C_\nu = \max_{q \geq 1} \left[\int_{\mathbb{R}_0} \mathbb{1}_{(|z| < 1)} |z|^q \nu(dz) + \int_{\mathbb{R}_0} \mathbb{1}_{(|z| \geq 1)} |z|^q \nu(dz) \right]$. Since $C_\beta := \sum_{k=1}^\infty \beta_k^\alpha < \infty$, $C_\nu < \infty$ for any $\alpha > 0$ and $q \geq 1$, respectively, we derive that there exists a constant $C > 0$ such that

$$\mathbb{E} \sup_{s \in [0, t]} |I_2(s)| \leq C t + C \mathbb{E} \int_0^t |\mathbf{u}_n(s)|^p ds.$$

Using Burkholder-Davis-Gundy inequality and similar idea as above, we deduce that

$$\begin{aligned} \mathbb{E} \sup_{s \in [0, t]} |I_1(s)| &\leq c_p \sum_{k=1}^n \mathbb{E} \left[\int_0^t \int_{|z| < 1} (|\mathbf{u}_n(t) + \beta_k z e_k|^p - |\mathbf{u}_n(t)|^p)^2 \nu(dz) dt \right]^{\frac{1}{2}} \\ &\leq C_p \sum_{k=1}^n \mathbb{E} \left[\int_0^t \int_{|z| < 1} \left(\beta_k^p |z|^p + \beta_k |z| |\mathbf{u}_n(t)|^{p-1} + \sum_{r=1}^{p-2} \beta_k^r |z|^r |\mathbf{u}_n(t)|^{p-r} \right)^2 \nu(dz) dt \right]^{\frac{1}{2}} \\ &\leq C_p \sum_{k=1}^n \mathbb{E} \left[\int_0^t \int_{|z| < 1} \left(\sum_{r=1}^p \beta_k^{2r} |z|^{2r} + \sum_{r=1}^{p-1} \beta_k^{2r} |z|^{2r} |\mathbf{u}_n(s)|^{2(p-1)} \right) \nu(dz) ds \right]^{\frac{1}{2}} \\ (16) \quad &\leq C_p C_\beta C_\nu t \left(1 + \mathbb{E} \sup_{s \in [0, t]} |\mathbf{u}_n(s)|^{p-1} \right). \end{aligned}$$

where we understand that $\sum_{r=2}^{p-1} \beta_k^r |z|^r |\mathbf{u}_n(t)|^{p-r} = 0$ if $p = 2$. Recall that for any real numbers $a \geq 0$ and $b \geq 0$ we have

$$ab \leq \frac{a^p}{p\varepsilon^p} + \frac{p-1}{p} (b\varepsilon)^{\frac{p}{p-1}},$$

for any $\varepsilon > 0$. We deduce from this and the inequality (16) that there exists $C > 0$ such that

$$\mathbb{E} \sup_{s \in [0, t]} |I_1(s)| \leq C t + \frac{1}{2} \mathbb{E} \sup_{s \in [0, t]} |\mathbf{u}_n(s)|^p.$$

Now we deal with $I_3(t)$. As above, it is easy to see that

$$||\mathbf{u}_n(t) + \beta_k z e_k|^p - |\mathbf{u}_n(t)|^p - \Psi_{\mathbf{u}_n, t}[\beta_k z e_k]| \leq C_p \left(\beta_k^p |z|^p + \sum_{r=1}^{p-1} \beta_k^r |z|^r |\mathbf{u}_n(t)|^{p-r} + p |\mathbf{u}_n|^{p-1} \beta_k |z| \right).$$

Thus, using Young's inequality we easily deduce that

$$||\mathbf{u}_n(t) + \beta_k z e_k|^p - |\mathbf{u}_n(t)|^p - \Psi_{\mathbf{u}_n, t}[\beta_k z e_k]| \leq C_p \left(\sum_{r=1}^p \beta_k^r |z|^r + \sum_{r=1}^{p-1} \beta_k^r |z|^r |\mathbf{u}_n(t)|^p \right).$$

Hence, arguing as in the case of I_2 we deduce that there exists a constant $C > 0$ such that

$$\mathbb{E} \sup_{s \in [0, t]} \left| \int_0^s I_3(r) dr \right| \leq C t + C \mathbb{E} \int_0^t |\mathbf{u}_n(s)|^p ds.$$

Summing up we have showed that there exists $C > 0$ such that for any $n \in \mathbb{N}$

$$\frac{1}{2} \mathbb{E} \sup_{s \in [0, t]} |\mathbf{u}_n(s)|^p + p \kappa \mathbb{E} \int_0^t |A^{\frac{1}{2}} \mathbf{u}_n(s)|^2 |\mathbf{u}_n(s)|^{p-2} ds \leq |\xi|^p + C t + C \mathbb{E} \int_0^t |\mathbf{u}_n(s)|^p ds.$$

Invoking the Gronwall's inequality we infer that there exists K_0 and K_1 such that for any $n \in \mathbb{N}$

$$(17) \quad \mathbb{E} \sup_{s \in [0, t]} |\mathbf{u}_n(s)|^p + 2p \kappa \mathbb{E} \int_0^t |\mathbf{u}_n(s)|^{p-2} |A^{\frac{1}{2}} \mathbf{u}_n(s)|^2 ds \leq (K_0 t + |\xi|^p) e^{K_1 t}.$$

Now, the existence of solution \mathbf{u} will follow from a similar argument as in [27] (see also [28], [36]). The uniqueness of the solution can be proved by arguing as in [27] or [10].

By Assumption 2.1, $\langle Ay, y \rangle = |A^{\frac{1}{2}}y|^2$ for any $y \in \mathbf{V}$, thus thanks to Assumption 2.2 we can argue as in [27] and show that for any $t \geq 0$ we have

$$(18) \quad \lim_{n \rightarrow \infty} \mathbb{E} |\mathbf{u}_n(t) - \mathbf{u}(t)|^2 = 0,$$

$$(19) \quad \lim_{n \rightarrow \infty} \mathbb{E} \int_0^t |A^{\frac{1}{2}}[\mathbf{u}_n(s) - \mathbf{u}(s)]|^2 ds = 0.$$

Now, we prove that \mathbf{u} has a càdlàg modification in \mathbf{H} . Our proof relies very much on recent result about càdlàg property of stochastic convolution proved in [39]. Let \mathfrak{S} be the stochastic convolution defined by

$$(20) \quad \mathfrak{S}(t) = \sum_{k=1}^{\infty} \mathfrak{S}_k(t) e_k,$$

where each \mathfrak{S}_k is the solution to

$$(21) \quad d\mathfrak{S}_k(t) = -\kappa \lambda_k \mathfrak{S}_k(t) dt + \beta_k \int_{\mathbb{R}_0} z d\bar{\eta}_k(z, t).$$

Since, by Assumption 2.1-(ii) and Assumption 2.10,

$$\sum_{k=1}^{\infty} (\lambda_k^{\varepsilon-1} \beta_k^2 + \lambda_k^{\varepsilon} \beta_k^4) \leq C \sum_{k=1}^{\infty} (\lambda^{-2k(\varepsilon-[1+2\theta])} + \lambda^{(\varepsilon-4\theta)2k}) < \infty,$$

for any $\varepsilon \in (0, 2)$, it follows from [39, Corollary 3.3] that \mathfrak{S} has a càdlàg modification in \mathbf{H} . Let us also consider the following problem

$$(22) \quad \frac{d\mathbf{v}(t)}{dt} + \kappa A\mathbf{v}(t) + \mathbf{B}(\mathbf{v}(t) + \mathfrak{S}(t), \mathbf{v}(t) + \mathfrak{S}(t)) = 0, \quad \mathbf{v}(0) = \xi \in \mathbf{H},$$

where $\mathfrak{S} \in L^\infty(0, T; \mathbf{H})$. Arguing as in Appendix C we can show that it has a unique solution $\mathbf{v} \in C(0, T; \mathbf{H}) \cap L^2(0, T; \mathbf{V})$. Taking \mathfrak{S} as the stochastic convolution defined in (20)-(21) we see, thanks to the uniqueness of solution, that $\mathbf{u} = \mathbf{v} + \mathfrak{S}$ solves (13). Thanks to [48, Theorem 4.1] the function $\phi : C([0, T]; \mathbf{H}) \times \mathbb{D}([0, T]; \mathbf{H}) \ni (x, y) \mapsto x + y \in \mathbb{D}([0, T]; \mathbf{H})$ is continuous, hence \mathbf{u} has a càdlàg modification in \mathbf{H} . Here $\mathbb{D}([0, T]; \mathbf{H})$ denotes the space, equipped with the Skorokhod topology J_1 , of càdlàg functions taking values in \mathbf{H} .

Arguing as in [2, Section 6] we can show that \mathbf{u} is a Markov semigroup. The idea in [27] can be used to prove that \mathbf{u} has the Feller property. \square

3.2. Uniqueness of the invariant measure for the stochastic Shell models. The preparatory result in the previous subsection enables us to define a Markov semigroup which is generated by the Markov solution \mathbf{u} to (13). More precisely, we can define a Markov semigroup as in the following definition.

Definition 3.4. Let $\{\mathcal{P}_t; t \geq 0\}$ be the Markov semigroup defined by

$$[\mathcal{P}_t \Phi](\xi) = \mathbb{E}[\Phi(\mathbf{u}(t, \xi))], \quad \Phi \in B_b(\mathbf{H}), \xi \in \mathbf{H}, t \geq 0,$$

where $\mathbf{u}(\cdot, \xi)$ is the unique solution to (13) with initial condition $\xi \in \mathbf{H}$. For simplicity we will write

$$\mathcal{P}_t \Phi(\xi) := [\mathcal{P}_t \Phi](\xi), \quad \Phi \in B_b(\mathbf{H}), \xi \in \mathbf{H}, t \geq 0,$$

throughout.

We will establish that the Markov semigroup $\{\mathcal{P}_t; t \geq 0\}$ has a unique invariant measure which then implies the ergodicity of the solution to (13).

First let us introduce an auxiliary problem. For this aim, let $R \in (0, \infty)$ and $\rho(\cdot) : [0, \infty) \rightarrow [0, 1]$ be a C^∞ and Lipschitz function such that

$$\rho(x) = \begin{cases} 1 & \text{if } x \in [0, 1], \\ 0 & \text{if } x \in [2, \infty], \\ \in [0, 1] & \text{if } x \in [1, 2], \end{cases}$$

and $|\rho'(x)| \leq 2$. For any $\mathbf{u} \in \mathbf{H}$ let $\{\mathbf{B}^R(\mathbf{u}, \mathbf{u}) : n \in \mathbb{N}\}$ be the family defined by

$$\mathbf{B}^R(\mathbf{u}, \mathbf{u}) := \rho\left(\frac{|\mathbf{u}|^2}{R}\right) \mathbf{B}(\mathbf{u}, \mathbf{u}), \text{ for any } R \in \mathbb{N}.$$

Let us consider the following modified problem

$$(23) \quad \begin{aligned} d\mathbf{u}^R(t) + [\kappa A\mathbf{u}^R(t) + \mathbf{B}^R(\mathbf{u}^R(t), \mathbf{u}^R(t))]dt &= \sum_{k=1}^{\infty} \beta_k dl_k(t) e_k, \\ \mathbf{u}^R(0) &= \xi \in \mathbf{H}. \end{aligned}$$

We have the following results which will be proved in the next section.

Proposition 3.5. *Let Assumption 2.1, Assumption 2.2, Assumption 2.10 and Assumption 2.3 hold. Then for each $R > 0$ and $\xi \in \mathbf{H}$ the problem (23) has a unique solution $\mathbf{u}^R := \mathbf{u}^R(\cdot, \xi)$. The stochastic process \mathbf{u}^R is a Markov process which has the Feller property. Furthermore, if $\{\mathcal{P}_t^R; t \geq 0\}$ denotes the Markov semigroup associated to \mathbf{u}^R (see Definition 3.4) then for any $R > 0$, $t > 0$ there exists a positive constant $C := C(t, R)$ such that*

$$(24) \quad |\mathcal{P}_t^R \Phi(\xi) - \mathcal{P}_t^R \Phi(\zeta)| < C \|\Phi\|_{\infty} |\xi - \zeta|,$$

for any $\xi, \zeta \in \mathbf{H}$ and $\Phi \in B_b(\mathbf{H})$.

Now, let us state two propositions whose proofs will be given below. The following proposition shows that the Markov semigroup associated to the unique solution of (13) has a certain smoothing property.

Proposition 3.6. *Let Assumption 2.1, Assumption 2.2, Assumption 2.10 and Assumption 2.3 hold. Let $\{\mathcal{P}_t; t \geq 0\}$, be the Markov process associated to the solution \mathbf{u} of (13). Then it has the strong Feller property, i.e., $\mathcal{P}_t B_b(\mathbf{H}) \subset C_b(\mathbf{H})$ for any $t > 0$.*

According to [25, Theorem 0.3], for the invariant measure to be unique it is sufficient to find a point $\xi \in \mathbf{H}$ that is accessible for \mathcal{P}_t . The definition of an accessible point is given in the following definition (see, for instance, [16], [25]).

Definition 3.7. Let \mathcal{R}_{λ} be the resolvent of \mathcal{P}_t defined by

$$\mathcal{R}_{\lambda}(\xi, \mathcal{U}) = \int_0^{\infty} e^{-\lambda t} [\mathcal{P}_t \mathbf{1}_{\mathcal{U}}](\xi) dt,$$

for any measurable set $\mathcal{U} \subset \mathbf{H}$, $\lambda > 0$ and $\xi \in \mathbf{H}$. A point $\mathbf{x} \in \mathbf{H}$ is accessible if, for every $\xi \in \mathbf{H}$ and every open neighborhood \mathcal{U} of \mathbf{x} , one has $\mathcal{R}_{\lambda}(\xi, \mathcal{U}) > 0$.

For our model we have the following result.

Proposition 3.8. *In addition to the assumptions of Proposition 3.6 suppose also that Assumption 2.8 holds. Then, the point $0 \in \mathbf{H}$ is accessible for $\{\mathcal{P}_t; t \geq 0\}$.*

Before we proceed to the statement and the proof of the main result of this paper we should give a refinement of the estimate (15) in Proposition 3.3.

Lemma 3.9. *There exists a constant $C > 0$ such that for any $T > 0$ and $\xi \in \mathbf{H}$ we have*

$$(25) \quad \mathbb{E}|\mathbf{u}(T, \xi)|^2 \leq (|\xi|^2 + CT)e^{-\frac{\kappa}{\lambda_1} T},$$

$$(26) \quad \mathbb{E} \int_0^T |A^{\frac{1}{2}} \mathbf{u}(s, \xi)|^2 ds \leq (|\xi|^2 + CT) + C \frac{\lambda_1}{\kappa}$$

Proof. We argue as in the proof of Proposition 3.3 and work with the Galerkin approximation. Note that thanks to the estimate (15) the stochastic process

$$M_n(t) = \sum_{k=1}^n \int_0^t \int_{|z| \leq 1} [\beta_k^2 |z|^2 + 2\beta_k \langle \mathbf{u}_n(s), e_k \rangle z] d\tilde{\eta}_k(z, s),$$

is a martingale satisfying $\mathbb{E}M_n(t) = 0$ for any $t > 0$. Therefore, arguing as before we derive the following chain of equalities/inequalities

$$\begin{aligned} \mathbb{E}|\mathbf{u}_n(t)|^2 + 2\kappa\mathbb{E}\int_0^t |A^{\frac{1}{2}}\mathbf{u}_n(s)|^2 ds &= |\xi|^2 + \sum_{k=1}^n \mathbb{E}\int_0^t \int_{|z|^2 > 1} [\beta_k^2 |z|^2 + 2\beta_k \langle \mathbf{u}_n(s), e_k \rangle z] d\eta_k(dz, ds) \\ &\quad + 2t \left(\int_{\mathbb{R}_0} |z|^2 \nu(dz) \right) \sum_{k=1}^n \beta_k^2 \\ &\leq |\xi|^2 + 4C_\beta C_\nu t + 2 \sum_{k=1}^n \int_0^t \int_{|z|^2 > 1} \mathbb{E}|\mathbf{u}_n(s)| |z| \nu(dz) ds \\ &\leq |\xi|^2 + 4C_\beta C_\nu t + 2C_\beta C_\nu \int_0^t \mathbb{E}|\mathbf{u}_n(s)| ds. \end{aligned}$$

Thanks to the Poincaré inequality (4) and the fact that $|\mathbf{u}_n(s)| \leq \frac{\lambda_1^2}{4\kappa} + \frac{\kappa}{\lambda_1^2} |\mathbf{u}_n(s)|^2$, we derive from the chain of inequalities above that

$$\mathbb{E}|\mathbf{u}_n(t)|^2 \leq |\xi|^2 + 2C_\beta C_\nu \left(2 + \frac{\lambda_1^2}{\kappa}\right) t - \frac{\kappa}{\lambda_1^2} \int_0^t \mathbb{E}|\mathbf{u}_n(s)|^2 ds,$$

which implies that

$$\mathbb{E}|\mathbf{u}_n(t)|^2 \leq (|\xi|^2 + 2C_\beta C_\nu \left(2 + \frac{\lambda_1^2}{\kappa}\right) t) e^{-\frac{\kappa}{\lambda_1^2} t}, \quad t > 0, \quad n \in \mathbb{N}.$$

Observe that

$$\mathbb{E}|\mathbf{u}_n(t)|^2 + 2\kappa\mathbb{E}\int_0^t |A^{\frac{1}{2}}\mathbf{u}_n(s)|^2 ds \leq |\xi|^2 + 4C_\beta C_\nu t + 2C_\beta C_\nu \int_0^t \mathbb{E}|\mathbf{u}_n(s)| ds.$$

Therefore, there exists $C > 0$ such that

$$\mathbb{E}\int_0^t |A^{\frac{1}{2}}\mathbf{u}_n(s)|^2 ds \leq C(|\xi|^2 + 1)t + C \int_0^t s e^{-\frac{\kappa}{\lambda_1^2} s} ds, \quad \forall t > 0, \quad n \in \mathbb{N}.$$

From the last two estimates, (18) and (19) we easily derive the proof of the lemma. \square

Now, we give in the next theorem the main result of the present work.

Theorem 3.10. *Let the assumptions of Proposition 3.8 holds. Then, the semigroup $\{\mathcal{P}_t; t \geq 0\}$ admits a unique invariant measure μ whose support is included in \mathbf{V} .*

Proof. Owing to the compact embedding $\mathbf{V} \subset \mathbf{H}$ and the estimates (25) and (26) the existence of an invariant measure μ follows from the Krylov–Bogolyubov theorem (see, for instance, the proofs in [13, Theorem 2.2] or [27, Theorem 5.3]). One can also argue as in [27, Theorem 5.3] to show that the support of μ is included in \mathbf{V} .

Thanks to Proposition 3.6 and Proposition 3.8 we infer from [25, Theorem 0.3] that the invariant measure μ is unique. \square

Before we proceed to the proofs of our results we state the following remark.

Remark 3.11. It is clear from Assumption 2.10 that the noise we consider in this paper is not cylindrical and it is not known whether our results hold for the stochastic shell models with cylindrical pure jump Lévy noise.

Now, we give the proofs of Proposition 3.6 and 3.8.

Proof of Proposition 3.6. We will show that for any $\xi \in \mathbf{H}$, $\varepsilon > 0$ there exists a constant $\delta > 0$ such that

$$|\mathcal{P}_t \Phi(\xi) - \mathcal{P}_t \Phi(\zeta)| \leq \varepsilon,$$

for any $\Phi \in B_b(\mathbf{H})$ with $\|\Phi\|_\infty \leq 1$, $\zeta \in B_{\mathbf{H}}(\xi, \delta)$.

For this purpose, let $\Phi \in B_b(\mathbf{H})$ and $R > R_0$ where R_0 will be fixed later. Let $\mathbf{u}(\cdot, \xi)$ and $\mathbf{u}(\cdot, \zeta)$ be solutions of (13) with the initial conditions $\xi \in \mathbf{H}$ and $\zeta \in \mathbf{H}$, respectively. For any $\xi \in \mathbf{H}$ let $\{\tau_R(\xi); R > 0\}$ be the family of stopping times defined by

$$\tau_R(\xi) := \inf\{t \geq 0; |\mathbf{u}(t, \xi)| \geq R\}.$$

Since, by definition of \mathbf{u}^R and uniqueness of solution of (13) (see also Remark 4.3), $\mathbf{u}(t, \cdot) = \mathbf{u}^R(t, \cdot)$ on $\{t \leq \tau_R(\cdot)\}$, we obtain that

$$\begin{aligned} |\mathcal{P}_t\Phi(\xi) - \mathcal{P}_t\Phi(\zeta)| &\leq |\mathcal{P}_t\Phi(\xi) - \mathcal{P}_t^R\Phi(\xi)| + |\mathcal{P}_t^R\Phi(\xi) - \mathcal{P}_t^R\Phi(\zeta)| + |\mathcal{P}_t^R\Phi(\zeta) - \mathcal{P}_t\Phi(\zeta)| \\ &\leq 2\|\Phi\|_\infty \mathbb{P}(\tau_R(\xi) < t) + 2\|\Phi\|_\infty \mathbb{P}(\tau_R(\zeta) < t) + |\mathcal{P}_t^R\Phi(\xi) - \mathcal{P}_t^R\Phi(\zeta)|. \end{aligned}$$

For any $t > 0$, $p \in \{2, 4\}$ and $R > 0$ we have

$$\mathbb{E}|\mathbf{u}(t \wedge \tau_R, \xi)|^p = \mathbb{E}(|\mathbf{u}(\tau_R, \xi)|^p \mathbf{1}_{\tau_R < t}) + \mathbb{E}(|\mathbf{u}(t, \xi)|^p \mathbf{1}_{t \leq \tau_R}).$$

From the càdlàg property of $\mathbf{u}(\cdot, \xi)$ and the definition of τ_R it follows that $\mathbb{E}|\mathbf{u}(\tau_R, \xi)|^p \geq R^p$. Thus,

$$\mathbb{E}|\mathbf{u}(t \wedge \tau_R, \xi)|^p \geq R^p \mathbb{P}(\tau_R < t).$$

By Proposition 3.3, for any $t > 0$ and $\xi \in \mathbf{H}$ there exists $C(t, \xi)$ such that

$$\mathbb{E} \sup_{s \in [0, t]} |\mathbf{u}(s, \xi)|^p < C(t, \xi),$$

from which and the former estimate we can deduce that for any $\xi \in \mathbf{H}$ and $t > 0$

$$(27) \quad \mathbb{P}(\tau_R(\xi) < t) \leq \frac{1}{R^p} C(t, \xi).$$

Thanks to Proposition 3.5 the Markov semigroup $\{\mathcal{P}_t^R, t \geq 0\}$ has the strong Feller property. In particular, we infer from (24) that for any $\Phi \in B_b(\mathbf{H})$, $\xi \in \mathbf{H}$ we have

$$|\mathcal{P}_t\Phi(\xi) - \mathcal{P}_t\Phi(\zeta)| \leq \|\Phi\|_\infty \left[\frac{4C(t, \xi)}{R^p} + C(R, t)|\xi - \zeta| \right].$$

Choosing

$$R \geq \left(\frac{8C(t, \xi)}{\varepsilon} \right)^{\frac{1}{p}} \text{ and } \delta \leq \frac{\varepsilon}{2C(R, t)},$$

we derive that

$$|\mathcal{P}_t\Phi(\xi) - \mathcal{P}_t\Phi(\zeta)| \leq \varepsilon,$$

for any $\xi \in \mathbf{H}$, $\zeta \in B_{\mathbf{H}}(\xi, \delta)$, and $\Phi \in B_b(\mathbf{H})$ with $\|\Phi\|_\infty \leq 1$. This proves that the Markov semigroup $\{\mathcal{P}_t; t \geq 0\}$ is strong Feller. \square

Proof of Proposition 3.8. The first step of the proof is to check the following claim. Fixed any $\varepsilon > 0$ and for any $t > 0$ define

$$\Omega^* = \{\omega \in \Omega : \sup_{s \in [0, t]} |\mathfrak{S}(s, \omega)|^2 < \varepsilon\}$$

where \mathfrak{S} is the stochastic convolution defined by (20)-(21).

Claim I: For any $\varepsilon > 0$ and $t > 0$ we have $\mathbb{P}(\Omega^*) > 0$.

To prove this claim we first observe that

$$\mathbb{P}(\Omega^*) \geq \mathbb{P}(\Omega_{1, N}^*) \cdot \mathbb{P}(\Omega_{2, N}^*),$$

where N is a certain large integer and

$$\Omega_{1, N}^* := \{\omega : \sup_{s \in [0, t]} \sum_{k \geq N} |\mathfrak{S}_k(s, \omega)|^2 < \frac{\varepsilon}{2}\},$$

$$\Omega_{2, N}^* := \{\omega : \sup_{s \in [0, t]} \sum_{k < N} |\mathfrak{S}_k(s, \omega)|^2 < \frac{\varepsilon}{2}\}.$$

Let

$$l_k(t) = \int_0^t \int_{\mathbb{R}} z \bar{\eta}_k(z, s), \quad t \geq 0, k \in \mathbb{N}.$$

In the remaining part of the proof we use without further notice the shorthand notations $\mathfrak{S}(s)$ and $l_k(s)$ to denote $\mathfrak{S}(s, \omega)$ and $l_k(s, \omega)$, respectively.

For each k the process l_k defines a Lévy process with Lévy measure ν . For each $t \geq 0$ and $k \in \mathbb{N}$ the function $e^{-\lambda_k(t-\cdot)}$ is differentiable, we can apply [40, Proposition 9.16] to derive that

$$\mathfrak{S}_k(t) = \left(l_k(t) - \kappa \lambda_k \int_0^t e^{-\lambda_k(t-s)} l_k(s) ds \right) \beta_k,$$

for any $t > 0$ and $k \in \mathbb{N}$. From this last identity we easily infer that

$$\sup_{s \in [0, t]} |\mathfrak{S}_k(s)| \leq (1 + \kappa) \beta_k \sup_{s \in [0, t]} |l_k(s)|,$$

for any $t > 0$ and $k \in \mathbb{N}$. Thanks to the inequality

$$\sup_{s \in [0, t]} \sum_{k \leq N} |\mathfrak{S}_k(s)|^2 \leq \sum_{k \leq N} \sup_{s \in [0, t]} |\mathfrak{S}_k(s)|^2 \leq (1 + \kappa)^2 \sum_{k \leq N} \beta_k^2 \sup_{s \in [0, t]} |l_k(s)|^2,$$

we have that

$$\mathbb{P}(\Omega_{2, N}^*) \geq \mathbb{P}\left(\sum_{k \leq N} \beta_k^2 \sup_{s \in [0, t]} |l_k(s)|^2 < \frac{\varepsilon}{2(1 + \kappa)^2}\right).$$

Since

$$\{\omega; \sup_{s \in [0, t]} |l_k(s)| < \frac{1 + \kappa}{\beta_k} \sqrt{\frac{\varepsilon}{2N}} \text{ for all } k \leq N\} \subset \{\omega; \sum_{k \leq N} \beta_k^2 \sup_{s \in [0, t]} |l_k(s)|^2 < \frac{\varepsilon}{2(1 + \kappa)^2}\}$$

and the elements of $\{l_k; k \leq N\}$ are mutually independent, we obtain that

$$\mathbb{P}(\Omega_{2, N}^*) \geq \mathbb{P}\left(\sum_{k \leq N} \beta_k^2 \sup_{s \in [0, t]} |l_k(s)|^2 < \frac{\varepsilon}{2(1 + \kappa)^2}\right) \geq \prod_{k \leq N} \mathbb{P}\left(\sup_{s \in [0, t]} |l_k(s)| < \frac{1 + \kappa}{\beta_k} \sqrt{\frac{\varepsilon}{2N}}\right).$$

Since the Lévy measure ν satisfies Assumption 2.8 we easily derive that $\mathbb{P}(\Omega_{2, N}^*) > 0$. Since the stochastic convolution \mathfrak{S} is càdlàg and taking values in \mathbf{H} it follows from [39, Theorem 2.3] that there exists an integer $N_0 > 0$ such that $\mathbb{P}(\Omega_{1, N}^*) > 0$ for any $N \geq N_0$. Now we easily conclude that $\mathbb{P}(\Omega_{1, N}^*) \cdot \mathbb{P}(\Omega_{2, N}^*) > 0$ and thus the proof of the **Claim I**.

Now we pass to the next step of the proof of Proposition 3.8. Before proceeding further we introduce a notation. For any fixed $\delta > 0$ and $T > 0$ set

$$\tilde{\Omega}^*(\delta, T) = \{\omega \in \Omega : \sup_{s \in [0, T]} |\mathfrak{S}(s, \omega)|^2 < \min\left(\delta, \frac{\kappa^2}{4\lambda_1 C_0^2}\right)\},$$

where C_0 is the positive constant from Assumption 2.2-(a). The next step of the proof is to check the validity of the following claim.

Claim II: *For any $R > 0$ and $\gamma > 0$ there exist $T_0 > 0$ and $\delta_0 > 0$ such that for any $t \geq T_0$, $\xi \in B_{\mathbf{H}}(R)$ and $\omega \in \tilde{\Omega}^*(\delta_0, T_0)$*

$$|\mathbf{u}(t, \omega)|^2 \leq \gamma.$$

To check this claim we closely follow [20]. We multiply (22) by \mathbf{v} in the scalar product of \mathbf{H} and obtain

$$\frac{1}{2} \frac{d}{dt} |\mathbf{v}(t)|^2 + \kappa |A^{\frac{1}{2}} \mathbf{v}(t)|^2 = \langle \mathbf{B}(\mathbf{v}(t), \mathfrak{S}(t)) + \mathbf{B}(\mathfrak{S}(t), \mathfrak{S}(t)), \mathbf{v}(t) \rangle,$$

where we used Assumption 2.2-(c). Using Assumption 2.2-(a) and Cauchy's inequality we derive that

$$\begin{aligned} \frac{d}{dt}|\mathbf{v}(t)|^2 + 2\kappa|A^{\frac{1}{2}}\mathbf{v}(t)|^2 &\leq 2\kappa^{-1}C_0^2[|\mathbf{v}(t)|^2|\mathfrak{S}(t)|^2 + |\mathfrak{S}(t)|^4] + \kappa|A^{\frac{1}{2}}\mathbf{v}(t)|^2 \\ \frac{d}{dt}|\mathbf{v}(t)|^2 &\leq 2\kappa^{-1}C_0^2[|\mathbf{v}(t)|^2|\mathfrak{S}(t)|^2 + |\mathfrak{S}(t)|^4] - \kappa|A^{\frac{1}{2}}\mathbf{v}(t)|^2. \end{aligned}$$

Using the inequality (4) we obtain

$$\frac{d}{dt}|\mathbf{v}(t)|^2 \leq [2\kappa^{-1}C_0^2|\mathfrak{S}(t)|^2 - \lambda^{-1}\kappa]|\mathbf{v}(t)|^2 + 2\kappa^{-1}C_0^2|\mathfrak{S}(t)|^4$$

Using the Gronwall's inequality we infer that on $\tilde{\Omega}^*(\delta, T)$ we have

$$|\mathbf{v}(t)|^2 \leq |\xi|^2 e^{-\frac{\kappa}{2\lambda_1}t} + 2\kappa^{-1}C_0^2 \left(\min \left(\delta, \frac{\kappa^2}{4\lambda_1 C_0^2} \right) \right)^2.$$

Thus, for any $R > 0$ and $\gamma > 0$ we can find T_0 and δ_0 such that on $\tilde{\Omega}^*(\delta_0, T_0)$

$$|\mathbf{v}(t)|^2 \leq \frac{\gamma}{4},$$

for any $t \geq T_0$ and $\xi \in B_{\mathbf{H}}(R)$. Choosing δ_0 small we can assume that on $\tilde{\Omega}^*(\delta_0, T_0)$

$$|\mathfrak{S}(t)|^2 \leq \frac{\gamma}{4}, \text{ for any } t \geq T_0.$$

Thus, for any $R > 0$ and $\gamma > 0$ we found $T_0 > 0$ and δ_0 such that on $\tilde{\Omega}^*(\delta_0, T_0)$

$$|\mathbf{u}(t)|^2 = |\mathbf{v}(t) + \mathfrak{S}(t)|^2 \leq \gamma,$$

for any $t \geq T_0$ and $\xi \in B_{\mathbf{H}}(R)$. This completes the proof of **Claim II**.

Now we finalize the proof of Proposition 3.8. We easily infer from **Claim II** that for any $R > 0$ and $\gamma > 0$ there exist two positive constants T_0, δ_0 such that for any $t \geq T_0$ and $\xi \in B_{\mathbf{H}}(R)$

$$[\mathcal{P}_t \mathbf{1}_{B_{\mathbf{H}}(\gamma)}](\xi) \geq \mathbb{P}(\tilde{\Omega}^*(\delta_0, T_0)).$$

Since, by **Claim I**, we know that $\mathbb{P}(\tilde{\Omega}^*(\delta_0, T_0)) > 0$ and $R > 0$ is arbitrary, we deduce that for any $t \geq T_0$ and $\xi \in \mathbf{H}$

$$[\mathcal{P}_t \mathbf{1}_{B_{\mathbf{H}}(\gamma)}](\xi) > 0.$$

This implies that for every $\xi \in \mathbf{H}$ and every open neighborhood \mathcal{U} of 0, one has $\mathcal{R}_\lambda(\xi, \mathcal{U}) > 0$, from which we conclude the proof of Proposition 3.8. \square

4. ANALYTIC STUDY OF THE MODIFIED STOCHASTIC SHELL MODEL (23)

In this section we analyze the modified stochastic shell model (23). We are mainly interested in the existence and uniqueness of the solution and its qualitative properties.

4.1. Resolvability of the modified problem (23). Let us first introduce the concept of solution.

Definition 4.1. Let $T > 0$ be a real number. An \mathbb{F} -adapted process \mathbf{u}^R taking values in \mathbf{H} is called a solution of Eq. (23) if the following conditions are satisfied:

- (i) $\mathbf{u}^R \in L^2(0, T; \mathbf{H})$ \mathbb{P} -almost surely,
- (ii) the following equality holds for every $t \in [0, T]$ and \mathbb{P} -a.s.,

$$(28) \quad (\mathbf{u}^R(t), \phi) = (\xi, \phi) - \int_0^t (\langle \kappa A \mathbf{u}^R(s) + \mathbf{B}^R(\mathbf{u}^R(s), \mathbf{u}^R(s)), \phi \rangle) ds + \langle L(t), \phi \rangle,$$

for any $\phi \in \mathbf{V}$.

Proposition 4.2. *Let the assumptions of Proposition 3.3 hold. Then for each $R > 0$ the problem (23) has a unique solution \mathbf{u}^R which has a càdlàg modification in \mathbf{H} . Moreover, \mathbf{u}^R is a Markov process having the Feller property.*

Proof. Let \mathbf{u}_n^R the solution of the following

$$(29) \quad \begin{aligned} d\mathbf{u}_n^R + [\kappa A \mathbf{u}_n^R + \mathbf{B}_n^R(\mathbf{u}_n^R, \mathbf{u}_n^R)]dt &= \sum_{k=1}^n \int_{\mathbb{R}_0} z \beta_k e_k d\bar{\eta}_k(dz, dt), \\ \mathbf{u}_n^R(0) &= \Pi_n \xi \in \mathbf{H}_n. \end{aligned}$$

The problem (29) is a system of SDEs with globally Lipschitz coefficients. Thus it has a unique càdlàg solution \mathbf{u}_n^R which is a Markov process taking values in \mathbf{H}_n (see, for instance, [1]).

Now the existence and uniqueness of a solution \mathbf{u}^R can be established by arguing exactly as in the proof of Proposition 3.3.

Note that thanks to Assumption 2.1 and Assumption 2.2 we can also argue as in [27] and show that for any $T > 0$ we have

$$(30) \quad \begin{aligned} \lim_{n \rightarrow \infty} \mathbb{E} |\mathbf{u}_n^R(T) - \mathbf{u}^R(T)|^2 &= 0, \\ \lim_{n \rightarrow \infty} \mathbb{E} \int_0^T |\Lambda^{\frac{1}{2}}[\mathbf{u}_n^R(s) - \mathbf{u}^R(s)]|^2 ds &= 0. \end{aligned}$$

Now it remains to prove that the solution \mathbf{u}^R to (23) has a càdlàg modification in \mathbf{H} . The argument is very similar to the idea of the proof of Proposition 3.3. Let \mathfrak{S} be the stochastic convolution defined in (20)-(21). Arguing as in Appendix C we can show that the following evolution equation

$$(31) \quad \begin{aligned} \frac{d}{dt} \mathbf{v}(t) + \kappa A \mathbf{v}(t) + \rho(|\mathbf{v}(t) + \mathfrak{S}(t)|^2/R) \mathbf{B}(\mathbf{v}(t) + \mathfrak{S}(t), \mathbf{v}(t) + \mathfrak{S}(t)) &= 0, \quad t \in [0, T], \\ \mathbf{v}(0) &= \xi \in \mathbf{H}, \end{aligned}$$

has a unique solution $\mathbf{v}^R \in C(0, T; \mathbf{H}) \cap L^2(0, T; \mathbf{V})$. Now, the stochastic process \mathbf{u}^R can be written as $\mathbf{u}^R = \mathbf{v}^R + \mathfrak{S}$ where $\mathbf{v}^R \in C(0, T; \mathbf{H}) \cap L^2(0, T; \mathbf{V})$ is the unique solution of (31). Thanks to [48, Theorem 4.1] the function

$$\phi : C([0, T]; \mathbf{H}) \times \mathbb{D}([0, T]; \mathbf{H}) \ni (x, y) \mapsto x + y \in \mathbb{D}([0, T]; \mathbf{H}),$$

is continuous, hence \mathbf{u}^R has a càdlàg modification in \mathbf{H} .

The proof that \mathbf{u}^R is a Markov process follows from the argument in [2, Section 6].

To show that \mathbf{u}^R has the Feller property we first remark that for any $\xi, \zeta \in \mathbf{H}$ we have

$$\begin{aligned} &|\mathbf{u}^R(t, \xi) - \mathbf{u}^R(t, \zeta)| - C e^{-\lambda_1 t} |\xi - \zeta| \\ &\leq C \int_0^t (t-s)^{-1/2} e^{-\lambda_1(t-s)} \|\mathbf{B}^R(\mathbf{u}^R(s, \xi), \mathbf{u}^R(s, \xi)) - \mathbf{B}^R(\mathbf{u}^R(s, \zeta), \mathbf{u}^R(s, \zeta))\|_{\mathbf{V}^*} ds. \end{aligned}$$

Since

$$\|\mathbf{B}^R(u, u) - \mathbf{B}^R(v, v)\|_{\mathbf{V}^*} \leq C_R |u - v|,$$

for any $u, v \in \mathbf{H}$ we infer that

$$|\mathbf{u}^R(t, \xi) - \mathbf{u}^R(t, \zeta)| \leq e^{-\lambda_1 t} |\xi - \zeta| + C \int_0^t (t-s)^{-1/2} e^{-\lambda_1(t-s)} |\mathbf{u}^R(s, \xi) - \mathbf{u}^R(s, \zeta)| ds.$$

From Gronwall's inequality we deduce that

$$|\mathbf{u}^R(t, \xi) - \mathbf{u}^R(t, \zeta)| \leq C(R, T) |\xi - \zeta|,$$

from which the Feller property of \mathbf{u}^R easily follows. \square

Remark 4.3. Let \mathbf{u} be the solution to (13) and $\{\tau_R; R \in \mathbb{N}\}$ be a sequence of stopping times defined by

$$\tau_R := \inf\{t \geq 0; |\mathbf{u}(t)| \geq R\}.$$

It is clear that $\mathbf{B}(\mathbf{u}(t), \mathbf{u}(t)) := \mathbf{B}^R(\mathbf{u}^R(t), \mathbf{u}^R(t))$ on $\{t \leq \tau_R\}$, thus by uniqueness of solution of the system (13) we infer that

$$\mathbf{u} = \mathbf{u}^R \text{ on } \{t \leq \tau_R\}.$$

4.2. Strong feller property of the solution of (29). We have seen in the proof of Theorem 4.2 that the solution \mathbf{u}_n^R of the Galerkin approximation (see equation (29)) generates a Markov semigroup $\mathcal{P}_{t,n}^R$ defined by

$$\mathcal{P}_{t,n}^R \Phi(\xi) = \mathbb{E}[\Phi(\mathbf{u}_n^R(t, \xi))],$$

for any $\Phi \in \mathcal{B}_b(\mathbf{H}_n)$ and $\xi \in \mathbf{H}_n$. Now, we will show the smoothing property of $\mathcal{P}_{t,n}^R$.

Since the coefficients of (29) belong to $C^2(\mathbf{H}_n; \mathbf{H}_n)$ the mapping $\xi_n \ni \mathbf{H}_n \mapsto \mathbf{u}_n^R$ is C^1 differentiable and the derivative $U_n^R(s, x) := \nabla_x \mathbf{u}_n^R(s, \xi)$ in the direction of $x \in \mathbf{H}_n$ at point $\xi_n \in \mathbf{H}_n$ is the solution of the linearized equation

$$(32) \quad \begin{aligned} dU_n^R(t, x) + [\kappa A U_n^R(t, x) + \nabla \mathbf{B}_n^R(\mathbf{u}_n^R(t, \xi), \mathbf{u}_n^R(t, \xi))][U_n^R(t, x)] dt &= 0, \\ U_n^R(0) &= x. \end{aligned}$$

Lemma 4.4. *For any $t > 0$ and $\xi \in \mathbf{H}_n$ the system (32) has a unique solution U_n^R such that $U_n^R \in C(0, t; \mathbf{H}_n) \cap L^2(0, t; \mathbf{V}_n)$. Moreover, for any $R > 0$ there exists a constant $C_R > 0$ such that*

$$(33) \quad \sup_{\substack{x \in \mathbf{H}_n, \\ |x| \leq 1}} \left[\mathbb{E}|U_n^R(s, x)|^2 + \kappa \mathbb{E} \int_0^t |A^{\frac{1}{2}} U_n^R(s, x)|^2 ds \right] \leq (1 + C_R e^{\frac{4}{\kappa} t}), \quad t > 0.$$

Proof. For the sake of simplicity we will write $U_n^R(\cdot) := U_n^R(\cdot, x)$. We will not dwell on the details of the existence of solution since it can be proved with standard argument. We will just prove the estimate (33). For this purpose, we start with the following identity

$$\nabla \mathbf{B}_n^R(\mathbf{u}_n^R, \mathbf{u}_n^R)[U_n^R] = \rho'_R(\mathbf{u}_n^R) \frac{\langle \mathbf{u}_n^R, U_n^R \rangle}{R|\mathbf{u}_n^R|} \mathbf{B}(\mathbf{u}_n^R, \mathbf{u}_n^R) + \rho_R(\mathbf{u}_n^R) [\mathbf{B}(\mathbf{u}_n^R, U_n^R) + \mathbf{B}(U_n^R, \mathbf{u}_n^R)].$$

Hence, it follows from Assumption 2.2-(a) and the definition of $\rho_R(\cdot)$ that

$$\begin{aligned} \|\nabla \mathbf{B}_n^R(\mathbf{u}_n^R, \mathbf{u}_n^R)[U_n^R]\|_{\mathbf{V}^*} &\leq \frac{1}{R} |\rho'(\mathbf{u}_n^R)| |U_n^R| \|\mathbf{B}(\mathbf{u}_n^R, \mathbf{u}_n^R)\|_{\mathbf{V}^*} + 2\rho_R(\mathbf{u}_n^R) \|\mathbf{B}(\mathbf{u}_n^R, U_n^R)\|_{\mathbf{V}^*} \\ &\leq \frac{C}{R} |\rho'(\mathbf{u}_n^R)| |U_n^R| |\mathbf{u}_n^R|^2 + C\rho_R(\mathbf{u}_n^R) |\mathbf{u}_n^R| |U_n^R| \\ &\leq C_R |U_n^R|. \end{aligned}$$

Therefore,

$$(34) \quad \begin{aligned} |\langle \nabla \mathbf{B}_n^R(\mathbf{u}_n^R, \mathbf{u}_n^R)[U_n^R], U_n^R \rangle| &\leq C_R |U_n^R| |A^{\frac{1}{2}} U_n^R| \\ &\leq C_R^2 \frac{2}{\kappa} |U_n^R|^2 + \frac{\kappa}{2} |A^{\frac{1}{2}} U_n^R|^2. \end{aligned}$$

Now multiplying (32) by $U_n^R(t)$ and plugging (34) in the resulting equation yields

$$\frac{1}{2} \frac{d}{dt} |U_n^R(t)|^2 + \kappa |A^{\frac{1}{2}} U_n^R(t)|^2 \leq C_R^2 \frac{2}{\kappa} |U_n^R(t)|^2 + \frac{\kappa}{2} |A^{\frac{1}{2}} U_n^R(t)|^2.$$

Thus,

$$\frac{d}{dt} |U_n^R(t)|^2 + \kappa |A^{\frac{1}{2}} U_n^R(t)|^2 \leq C_R^2 \frac{4}{\kappa} |U_n^R(t)|^2,$$

from which along with the Gronwall inequality we infer that

$$|U_n^R(t)|^2 + \kappa \int_0^t |A^{\frac{1}{2}} U_n^R(s)|^2 ds \leq |U_n^R(0)|^2 (1 + C_R e^{\frac{4}{\kappa} t}).$$

We easily conclude the proof from this last inequality. \square

We have the following lemma.

Lemma 4.5. *Suppose that all the assumptions of Proposition 3.5 are verified. Then, for any $R > 0$, $t > 0$ there exists a positive constant $C := C(t, R)$ such that*

$$(35) \quad |\mathcal{P}_{t,n}^R \Phi(\xi) - \mathcal{P}_{t,n}^R \Phi(\zeta)| < C \|\Phi\|_\infty |\xi - \zeta|,$$

for any $n \in \mathbb{N}$, $\xi, \zeta \in \mathbf{H}_n$ and $\Phi \in B_b(\mathbf{H}_n)$.

Proof. The idea is to use the estimate for the gradient of the Markovian semigroup $\mathcal{P}_{t,n}^R$. Let $\Phi \in C_b^2(\mathbf{H}_n)$ and $\nabla_x \mathcal{P}_{t,n}^R \Phi(\xi)$ be the derivative in the direction of $x \in \mathbf{H}_n$ at a point $\xi \in \mathbf{H}_n$ of $\mathcal{P}_{t,n}^R \Phi(\cdot)$. Notice that when identifying \mathbf{H}_n with \mathbb{R}^n the linear operator A^δ , $\delta \in [0, \infty)$, can be identified with the diagonal matrix $[A_{jk}^\delta; j, k = 1, \dots, n]$ defined by

$$A_{jk}^\delta = \begin{cases} \lambda_j^\delta & \text{if } j = k \\ 0 & \text{otherwise.} \end{cases}$$

Thanks to the estimate (51) in Lemma B.1 we have

$$\sup_{\substack{n \in \mathbb{N}, \xi, x \in \mathbf{H}_n \\ |x| \leq 1}} |\nabla_x \mathcal{P}_{t,n}^R \Phi(\xi)| \leq C(t) \left(\sum_{j=1}^{\infty} \beta_j^{-2} \lambda_j^{-1} \right)^{\frac{1}{2}} \|\Phi\|_\infty \left[\mathbb{E} \int_0^t |A^{\frac{1}{2}} U_n^R(s)|^2 ds \right]^{\frac{1}{2}},$$

where we have used the shorthand notation $U_n^R(\cdot) := U_n^R(\cdot, x)$. Owing to (33) we obtain the following estimate

$$\sup_{\substack{n \in \mathbb{N}, \xi, x \in \mathbf{H}_n \\ |x| \leq 1}} |\nabla_x \mathcal{P}_{t,n}^R \Phi(\xi)| \leq C(t) \left(\sum_{j=1}^{\infty} \beta_j^{-2} \lambda_j^{-1} \right)^{\frac{1}{2}} \|\Phi\|_\infty (1 + C R e^{\frac{4}{\kappa} t}).$$

Now we easily derive that for any $R > 0$, $t > 0$ there exists a constant $C := C(t, R) > 0$ such that

$$\sup_{\substack{n \in \mathbb{N}, \xi, x \in \mathbf{H}_n \\ |x| \leq 1}} |\nabla_x \mathcal{P}_{t,n}^R \Phi(\xi)| \leq C(t, R) \|\Phi\|_\infty,$$

for any $\Phi \in C_b^2(\mathbf{H}_n)$. Now we easily see that the estimate (35) holds for $\Phi \in C_b^2(\mathbf{H}_n)$. Owing to the equivalence lemma [40, Lemma 2.2] it follows that (35) also holds for $\Phi \in B_b(\mathbf{H}_n)$, and this completes the proof of our claim. \square

4.3. Strong Feller property of the solution to (23). In this section we will prove that for any $R > 0$ the semigroup \mathcal{P}_t^R associated to the solution \mathbf{u}^R of the modified problem (23) has the strong Feller property.

Proposition 4.6. *Suppose that all the assumptions of Proposition 3.5 are satisfied. Then, for any $R > 0$, $t > 0$ there exists a positive constant $C := C(t, R)$ such that*

$$(36) \quad |\mathcal{P}_t^R \Phi(\xi) - \mathcal{P}_t^R \Phi(\zeta)| < C \|\Phi\|_\infty |\xi - \zeta|,$$

for any $\xi, \zeta \in \mathbf{H}$ and $\Phi \in B_b(\mathbf{H})$.

Proof. Since, by (61), \mathbf{u}_n^R converges to \mathbf{u}^R strongly in $L^2(0, t; H)$ \mathbb{P} -a.s. we can infer the existence of a subsequence n_k such that

$$\mathbf{u}_{n_k}^R \rightarrow \mathbf{u}^R \quad dt \times d\mathbb{P} - \text{almost everywhere.}$$

Thanks to this convergence, the continuity and the boundedness of Φ we can derive from the Lebesgue Dominated Convergence Theorem that as $n_k \rightarrow \infty$

$$\mathbb{E} \int_0^t |\Phi(\mathbf{u}_{n_k}^R(t, \xi)) - \Phi(\mathbf{u}^R(t, \xi))| ds \rightarrow 0.$$

Hence there exists a subsequence, denoted again by n_k , such that

$$\mathbb{E}[\Phi(\mathbf{u}_{n_k}^R(s, \xi))] \rightarrow \mathbb{E}[\Phi(\mathbf{u}^R(s, \xi))]$$

for almost all $s \in [0, t]$. Hence thanks to this last convergence and (35) there exists $I_0 \subset (0, t]$ with $\text{Leb}(I_0) = 0$ such that for any $R > 0$ and $s \in I_0$ we have

$$(37) \quad |\mathcal{P}_s^R \Phi(\xi) - \mathcal{P}_s^R \Phi(\zeta)| \leq C_{R,s} \|\Phi\|_\infty |\xi - \zeta|,$$

for any $\xi, \zeta \in \mathbf{H}$ and $\Phi \in C_b^2(\mathbf{H})$. Since, by Proposition 4.2, \mathbf{u}^R is càdlàg in \mathbf{H} , the function $s \mapsto \mathcal{P}_s^R \Phi(\xi) - \mathcal{P}_s^R \Phi(\zeta)$, for any $\xi, \zeta \in \mathbf{H}$ and $\Phi \in C_b^2(\mathbf{H})$, is also càdlàg and the estimate (37) holds for all $s \in (0, t]$. This ends the proof of our claim. \square

APPENDIX A. BISMUT-ELWORTHY-LI FORMULA

In this first appendix we give and prove a Bismut-Elworthy-Li type formula for stochastic differential equations driven by pure jump noise. The proof is mainly a modification of [45, Proof of Theorem 1]. We will also follow closely the notation in [45].

Let $\eta := (\eta_1, \dots, \eta_n)$ be a Poisson random measure on \mathbb{R}^n , where η_1, \dots, η_n are n -independent Poisson random measure with Lévy measures ν_1, \dots, ν_n on $\mathbb{R}_0 := \mathbb{R} \setminus \{0\}$. The Lévy measure of η is denoted by $\nu(dz) := (\nu_1(dz_1), \dots, \nu_n(dz_n))$. We use the symbol

$$(\hat{\eta}_1(z_1, t), \dots, \hat{\eta}_n(z_n, t)) := (\nu_1(dz_1)dt, \dots, \nu_n(dz_n)dt),$$

to denote the compensator of η . The symbol $\tilde{\eta} = (\tilde{\eta}_1, \dots, \tilde{\eta}_n)$ describes the compensated Poisson random measures associated to η . To shorten notation we will use the following shorthand notations $d\eta(z, t) := \eta(dz, dt)$, $d\tilde{\eta}(z, t) := \tilde{\eta}(dz, dt)$ and $d\nu(z)dt := \nu(dz)dt$. We will also use the notation $d\bar{\eta}(z, t) := (d\bar{\eta}_1(z_1, t), \dots, d\bar{\eta}_n(z_n, t))$ where

$$d\bar{\eta}_j(z_j, t) = \mathbb{1}_{|z_j| \leq 1} d\tilde{\eta}_j(z_j, t) + \mathbb{1}_{|z_j| > 1} d\eta_j(z_j, t).$$

For this appendix we impose the following sets of conditions.

Assumptions A.1. (1) For each j there exists a C^1 function $g_j : \mathbb{R} \rightarrow [0, \infty)$ such that $\nu_j(dz_j) = g_j(z_j)dz_j$.

(2) As $|z| \rightarrow \infty$ we have $z^2 g_j(z) \rightarrow 0$ for each j . Also

$$\int_{\mathbb{R}_0} \left[|z|^q \mathbb{1}_{(|z| \leq 1)} + |z|^q \mathbb{1}_{(|z| > 1)} \right] \nu_j(dz) < \infty,$$

for any $q \geq 1$ and $j \in \{1, \dots, n\}$.

(3) Furthermore, there exists a constant $\alpha > 0$ such that for any $j \in \{1, \dots, n\}$ and $y \in \mathbb{R}$

$$\liminf_{\varepsilon \searrow 0} \varepsilon^\alpha \int_{\mathbb{R}_0} (|z \cdot y / \varepsilon|^2 \wedge 1) \nu_j(dz) > 0.$$

Assumptions A.2. Let $\alpha : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be a nonlinear map such that $\alpha(\cdot)$ belongs to $C_b^2(\mathbb{R}^n, \mathbb{R}^n)$.

Let γ be the $n \times n$ diagonal matrix given by

$$\gamma_{ij}(z) = \begin{cases} \beta_i z_i & \text{if } i = j, \\ 0 & \text{otherwise,} \end{cases}$$

where $\{\beta_i; i = 1, \dots, n\}$ is a family of positive numbers.

Let $\mathbf{X}(t, x) := (\mathbf{X}^{(1)}(t, x), \dots, \mathbf{X}^{(n)}(t, x))$ be the unique solution to the system of n SDEs given by

$$d\mathbf{X}(t) = \alpha(\mathbf{X}(t))dt + \int_{\mathbb{R}_0^n} \gamma(z) d\bar{\eta}(z, t), \quad \mathbf{X}(0) = x \in \mathbb{R}^n$$

that is

$$(38) \quad d\mathbf{X}^{(i)}(t) = \alpha^{(i)}(\mathbf{X}(t))dt + \sum_{j=1}^n \int_{\mathbb{R}_0} \gamma_{ij}(z_j) d\bar{\eta}_j(z_j, t), \quad \mathbf{X}^{(i)}(0) = x_i \quad i = 1, \dots, n.$$

Note that for any $x \in \mathbf{R}^n$ the process $\mathbf{X}(\cdot, x)$ is a Markov process. Thanks to Assumption A.2, it is proved in [45] that the map $\mathbb{R}^n \ni x \mapsto \mathbf{X}(t)$ has a C^1 -modification and its Jacobi matrix $U(t) := [U_{kj}(t); k, j \in \{1, \dots, n\}] = [\frac{\partial \mathbf{X}^{(j)}(t)}{\partial x_k}; k, j \in \{1, \dots, n\}]$ satisfies

$$\frac{d}{dt}U(t) = \nabla_x \alpha(\mathbf{X}(t, x))U(t), \quad U(0) = I_n$$

Let $\Lambda(s, z)$ be the matrix defined by

$$\Lambda_{kj}(s, z) = z_j^2 g_j(z_j) \beta_j^{-1} \frac{\partial \mathbf{X}^{(j)}(s, x)}{\partial x_k},$$

and $J(t) := (J^{(1)}(t), \dots, J^{(n)}(t))$ be the vector defined by

$$J^{(k)}(t) = \sum_{j=1}^n \int_0^t \int_{\mathbb{R}_0} \frac{1}{g_j(z_j)} \frac{\partial \Lambda_{kj}(s, z)}{\partial z_j} d\tilde{\eta}_j(z_j, s).$$

For $t > 0$ we set

$$K^{(k)}(t) = -2 \sum_{j=1}^n \int_0^t \int_{\mathbb{R}_0} z_j \beta_j^{-1} \frac{\partial \mathbf{X}^{(j)}(s, x)}{\partial x_k} d\eta_j(z_j, s),$$

and

$$\mathcal{A}(t) = \sum_{j=1}^n \int_0^t \int_{\mathbb{R}_0} z_j^2 d\eta_j(z_j, s).$$

Lemma A.3. *Let Assumption A.1 and Assumption A.2 hold. Then*

$$\nabla_{x_k} \mathbb{E}[\Phi(\mathbf{X}(t, x))] = \mathbb{E} \left[\Phi(\mathbf{X}(t, x)) \frac{K^{(k)}(t)}{[\mathcal{A}(t)]^2} \right] - \mathbb{E} \left[\Phi(\mathbf{X}(t, x)) \frac{J^{(k)}(t)}{\mathcal{A}(t)} \right],$$

for any $\Phi \in C_b^2(\mathbb{R}^n)$, $x \in \mathbb{R}^n$ and $k \in \{1, \dots, n\}$.

Proof. Our proof is mainly based on the arguments in [45, Section 4, Proof of Theorem 1] to which we refer for the omitted details. Here we only dwell on the parts where our idea and the arguments in [45] differ.

Let $u(s, x) := \mathbb{E}[\Phi(\mathbf{X}(t-s, x)) | \mathbf{X}(0) = x]$ for any $s \in [0, t]$ and $\Phi \in C_b^2(\mathbb{R}^n)$. Let $\gamma^{(j)}$ be the j -th column of the matrix γ and for any $y \in \mathbb{R}^n$ let

$$\mathcal{B}_z^{(j)} f(y) := f(y + \gamma^{(j)}(z)) - f(y).$$

Arguing as in [45, Lemma 4.1] and [45, Lemma 4.2] respectively, we can show that

$$(39) \quad \Phi(\mathbf{X}(t, x)) = \mathbb{E}[\Phi(\mathbf{X}(t, x))] + \sum_{j=1}^n \int_0^t \int_{\mathbb{R}_0} \mathcal{B}_z^{(j)} u(s, \mathbf{X}(s-, x)) d\tilde{\eta}_j(z_j, s),$$

and

$$(40) \quad \nabla_{x_k} \Phi(\mathbf{X}(t, x)) = \mathbb{E}[\nabla_{x_k} \Phi(\mathbf{X}(t, x))] + \sum_{j=1}^n \int_0^t \int_{\mathbb{R}_0} \nabla_{x_k} \mathcal{B}_z^{(j)} u(s, \mathbf{X}(s-, x)) d\tilde{\eta}_j(z_j, s),$$

for any $k \in \{1, \dots, n\}$.

Hereafter we fix $k, j \in \{1, \dots, n\}$. Since, by Assumption A.1-(2), $\int_{\mathbb{R}_0} z^2 \nu_j(dz_j) < \infty$ for any $j = 1, \dots, n$, we can use the same argument as in [45, Proof of Lemma 4.6] to prove that

$$(41) \quad \mathbb{E} \left[\sum_{j=1}^n \int_0^t \int_{\mathbb{R}_0} \mathcal{B}_z^{(j)} u(s, \mathbf{X}(s-, x)) d\tilde{\eta}_j(z_j, s) \times \sum_{j=1}^n \int_0^t \int_{\mathbb{R}_0} z_j^2 d\tilde{\eta}_j \right] = \sum_{j=1}^n \mathbb{E} \left[\Phi(\mathbf{X}(t, x)) \int_0^t \int_{\mathbb{R}_0} z_j^2 d\tilde{\eta}_j(z_j, s) \right].$$

In fact, if we replace the right hand side (RHS) $\Phi(\mathbf{X}(t, x))$ by the RHS of (39) and take into account that

$$\mathbb{E} \left[\mathbb{E}[\Phi(\mathbf{X}(t, x))] \int_0^t \int_{\mathbb{R}_0} z_j^2 d\tilde{\eta}_j(z_j, s) \right] = 0,$$

then (41) follows from the Itô formula. Similarly, we can use equation (40), to show first

$$(42) \quad \begin{aligned} \mathbb{E} \left[\sum_{j=1}^n \int_0^t \int_{\mathbb{R}_0} \nabla_{x_k} \mathcal{B}_z^{(j)} u(s, \mathbf{X}(s-, x)) d\tilde{\eta}_j(z_j, s) \times \sum_{j=1}^n \int_0^t \int_{\mathbb{R}_0} z_j^2 d\tilde{\eta}_j(z_j, s) \right] \\ = \sum_{j=1}^n \mathbb{E} \left[\nabla_{x_k} \Phi(\mathbf{X}(t, x)) \int_0^t \int_{\mathbb{R}_0} z_j^2 d\tilde{\eta}_j(z_j, s) \right], \end{aligned}$$

and, secondly, by Itô's formula

$$(43) \quad \begin{aligned} \mathbb{E} \left[\sum_{j=1}^n \int_0^t \int_{\mathbb{R}_0} \nabla_{x_k} \mathcal{B}_z^{(j)} u(s, \mathbf{X}(s-, x)) d\tilde{\eta}_j(z_j, s) \times \sum_{j=1}^n \int_0^t \int_{\mathbb{R}_0} z_j^2 d\tilde{\eta}_j(z_j, s) \right] \\ = \sum_{j=1}^n \mathbb{E} \left[\int_0^t \int_{\mathbb{R}_0} \nabla_{x_k} \mathcal{B}_z^{(j)} u(s, \mathbf{X}(s-, x)) z_j^2 d\nu_j(z_j) ds \right]. \end{aligned}$$

Hence, plugging this last identity, i.e. (43), in (42) yields

$$(44) \quad \mathbb{E} \left[\nabla_{x_k} \Phi(\mathbf{X}(t, x)) \times \sum_{j=1}^n \int_0^t \int_{\mathbb{R}_0} z_j^2 d\tilde{\eta}_j(z_j, s) \right] = \sum_{j=1}^n \mathbb{E} \left[\int_0^t \int_{\mathbb{R}_0} \nabla_{x_k} \mathcal{B}_z^{(j)} u(s, \mathbf{X}(s-, x)) z_j^2 d\nu_j(z_j) ds \right].$$

In the other hand, since $u(s, \mathbf{X}(s, x)) = \mathbb{E}[\Phi(\mathbf{X}(t-s, y)) | y = X(s, x)]$, we easily deduce from the Markov and the tower property of mathematical expectation that

$$\mathbb{E} u(s, \mathbf{X}(s, x)) = \mathbb{E} [\mathbb{E}[\Phi(\mathbf{X}(t-s, y)) | y = X(s, x)]] = \mathbb{E} [\Phi(\mathbf{X}(t, x))].$$

Thus, we infer from Fubini's theorem that

$$\begin{aligned} \sum_{j=1}^n \mathbb{E} \int_0^t \int_{\mathbb{R}_0} u(s, \mathbf{X}(s, x)) z_j^2 d\nu_j(z_j) ds \\ = \sum_{j=1}^n \int_0^t \int_{\mathbb{R}_0} \mathbb{E} [\mathbb{E}[\Phi(\mathbf{X}(t-s, y)) | y = \mathbf{X}(s, x)]] z_j^2 d\nu_j(z_j) ds \\ = \sum_{j=1}^n \int_0^t \int_{\mathbb{R}_0} \mathbb{E} \Phi(\mathbf{X}(t, x)) z_j^2 d\nu_j(z_j) ds \\ = \mathbb{E} \left[\Phi(\mathbf{X}(t, x)) \times \sum_{j=1}^n \int_0^t \int_{\mathbb{R}_0} z_j^2 d\nu_j(z_j) ds \right]. \end{aligned}$$

In the very same way we get

$$\mathbb{E} \left[\nabla_{x_k} \Phi(\mathbf{X}(t, x)) \times \sum_{j=1}^n \int_0^t \int_{\mathbb{R}_0} z_j^2 d\nu_j(z_j) ds \right] = \sum_{j=1}^n \mathbb{E} \left[\int_0^t \int_{\mathbb{R}_0} \nabla_{x_k} u(s, \mathbf{X}(s-, x)) z_j^2 d\nu_j(z_j) ds \right].$$

Now, observe that we have

$$(45) \quad \nabla_{x_k} \mathcal{B}_z^{(j)} u(s, \mathbf{X}(s-, x)) = \nabla_{x_k} u(s, \mathbf{X}(s-, x) + \gamma^{(j)}(z)) - \nabla_{x_k} u(s, \mathbf{X}(s-, x)),$$

and

$$(46) \quad \int_0^t \int_{\mathbb{R}_0} z_j^2 d\tilde{\eta}_j(z_j, s) = \int_0^t \int_{\mathbb{R}_0} z_j^2 d\tilde{\eta}_j(z_j, s) + \int_0^t \int_{\mathbb{R}_0} z_j^2 d\nu_j(z_j) ds,$$

Therefore, using (46) in one hand yields

$$\begin{aligned} & \mathbb{E} \left[\nabla_{x_k} \Phi(\mathbf{X}(t, x)) \times \sum_{j=1}^n \int_0^t \int_{\mathbb{R}_0} z_j^2 d\eta_j(z_j, s) \right] \\ &= \mathbb{E} \left[\nabla_{x_k} \Phi(\mathbf{X}(t, x)) \times \sum_{j=1}^n \int_0^t \int_{\mathbb{R}_0} z_j^2 d\tilde{\eta}_j(z_j, s) \right] + \mathbb{E} \left[\nabla_{x_k} \Phi(\mathbf{X}(t, x)) \times \sum_{j=1}^n \int_0^t \int_{\mathbb{R}_0} z_j^2 d\nu_j(z_j, s) \right], \end{aligned}$$

and in the other hand, using (45) we derive from (44) that

$$\begin{aligned} & \mathbb{E} \left[\nabla_{x_k} \Phi(\mathbf{X}(t, x)) \times \sum_{j=1}^n \int_0^t \int_{\mathbb{R}_0} z_j^2 d\eta_j(z_j, s) \right] \\ &= \sum_{j=1}^n \mathbb{E} \left[\int_0^t \int_{\mathbb{R}_0} \nabla_{x_k} \mathcal{B}_z^{(j)} u(s, \mathbf{X}(s-, x)) z_j^2 d\nu_j(z_j) ds \right] + \mathbb{E} \left[\nabla_{x_k} \Phi(\mathbf{X}(t, x)) \times \sum_{j=1}^n \int_0^t \int_{\mathbb{R}_0} z_j^2 d\nu_j(z_j, s) \right] \\ &= \sum_{j=1}^n \mathbb{E} \left[\int_0^t \int_{\mathbb{R}_0} \nabla_{x_k} u(s, \mathbf{X}(s-, x) + \gamma^{(j)}(z)) z_j^2 d\nu_j(z_j) ds \right]. \end{aligned}$$

That is,

$$\begin{aligned} (47) \quad & \mathbb{E} \left[\nabla_{x_k} \Phi(\mathbf{X}(t, x)) \times \sum_{j=1}^n \int_0^t \int_{\mathbb{R}_0} z_j^2 d\eta_j(z_j, s) \right] \\ &= \sum_{j=1}^n \mathbb{E} \left[\int_0^t \int_{\mathbb{R}_0} \nabla_{x_k} u(s, \mathbf{X}(s-, x) + \gamma^{(j)}(z)) z_j^2 d\nu_j(z_j) ds \right]. \end{aligned}$$

Next, by the same argument as used to show formula (41) we obtain, (see also [45, Proof of Lemma 4.6])

$$\begin{aligned} & \mathbb{E}[\Phi(\mathbf{X}(t, x)) J^{(k)}(t)] \\ &= \mathbb{E} \left[\sum_{j=1}^n \int_0^t \int_{\mathbb{R}_0} \mathcal{B}_z^{(j)} u(s, \mathbf{X}(s-, x)) d\tilde{\eta}_j(z_j, s) \times \sum_{j=1}^n \int_0^t \int_{\mathbb{R}_0} \frac{1}{g_j(z_j)} \frac{\partial \Lambda_{kj}(s, z)}{\partial z_j} d\tilde{\eta}_j(z_j, s) \right]. \end{aligned}$$

Continuing, and using the definition of g_j we get

$$\begin{aligned} & \mathbb{E}[\Phi(\mathbf{X}(t, x)) J^{(k)}(t)] \\ &= \sum_{j=1}^n \mathbb{E} \left[\int_0^t \int_{\mathbb{R}_0} \mathcal{B}_z^{(j)} u(s, \mathbf{X}(s-, x)) \frac{1}{g_j(z_j)} \frac{\partial \Lambda_{kj}(s, z)}{\partial z_j} d\nu_j(dz_j) ds \right] \\ &= \sum_{j=1}^n \mathbb{E} \left[\int_0^t \int_{\mathbb{R}_0} \mathcal{B}_z^{(j)} u(s, \mathbf{X}(s-, x)) \frac{\partial \Lambda_{kj}(s, z)}{\partial z_j} dz_j ds \right], \end{aligned}$$

By integration-by-parts, using the Assumption A.1-(2), and recalling the definition of Λ_{kj} we derive that

$$\begin{aligned} (48) \quad & \mathbb{E}[\Phi(\mathbf{X}(t, x)) J^{(k)}(t)] = - \sum_{j=1}^n \mathbb{E} \left[\int_0^t \int_{\mathbb{R}_0} \frac{\partial}{\partial z_j} \mathcal{B}_z^{(j)} u(s, \mathbf{X}(s-, x)) \Lambda_{kj} dz_j ds \right] \\ &= - \sum_{j=1}^n \mathbb{E} \left[\int_0^t \int_{\mathbb{R}_0} \frac{\partial}{\partial z_j} \mathcal{B}_z^{(j)} u(s, \mathbf{X}(s-, x)) z_j^2 \beta_j^{-1} \frac{\partial \mathbf{X}^{(j)}(s-, x)}{\partial x_k} g_j(z_j) dz_j ds \right] \\ &= - \sum_{j=1}^n \mathbb{E} \left[\int_0^t \int_{\mathbb{R}_0} \frac{\partial}{\partial z_j} \mathcal{B}_z^{(j)} u(s, \mathbf{X}(s-, x)) z_j^2 \beta_j^{-1} \frac{\partial \mathbf{X}^{(j)}(s-, x)}{\partial x_k} d\nu_j(z_j) ds \right]. \end{aligned}$$

We have the following partial derivatives identities

$$\frac{\partial}{\partial z_j} \mathcal{B}_z^{(j)} u(s, \mathbf{X}(s-, x)) = \frac{\partial}{\partial y_j} u^{(j)}(s, y) \Big|_{y=\mathbf{X}(s-, x) + \gamma^{(j)}(z)} \frac{\partial \gamma^{(j)}}{\partial z_j} = \beta_j \frac{\partial}{\partial y_j} u^{(j)}(s, y) \Big|_{y=\mathbf{X}(s-, x) + \gamma^{(j)}(z)}$$

and

$$\begin{aligned} \nabla_{x_k} u^{(j)}(s, \mathbf{X}(s-, x) + \gamma^{(j)}(z)) &= \frac{\partial u^{(j)}(s, \mathbf{X}(s-, x) + \gamma^{(j)}(z))}{\partial x_k} \\ &= \frac{\partial}{\partial y_j} u^{(j)}(s, y) \Big|_{y=\mathbf{X}(s-, x) + \gamma^{(j)}(z)} \frac{\partial \mathbf{X}^{(j)}(s-, x)}{\partial x_k}. \end{aligned}$$

Hence,

$$\beta_j^{-1} \frac{\partial}{\partial z_j} \mathcal{B}_z^{(j)} u(s, \mathbf{X}(s-, x)) \frac{\partial}{\partial x_k} \mathbf{X}^{(j)}(s, x) = \nabla_{x_k} u^{(j)}(s, \mathbf{X}(s-, x) + \gamma^{(j)}(z)).$$

Using this identity in (48) implies

$$\mathbb{E}[\Phi(\mathbf{X}(t, x) J^{(k)}(t))] = - \sum_{j=1}^n \mathbb{E} \left[\int_0^t \int_{\mathbb{R}_0} \nabla_{x_k} u(s, \mathbf{X}(s-, x) + \gamma^{(j)}(z)) z_j^2 d\nu_j(z_j) ds \right],$$

from which along with the identity (47) we derive that

$$(49) \quad \mathbb{E}[\Phi(\mathbf{X}(t, x) J^{(k)}(t))] = - \mathbb{E} \left[\nabla_{x_k} \Phi(\mathbf{X}(t, x)) \times \sum_{j=1}^n \int_0^t \int_{\mathbb{R}_0} z_j^2 d\eta_j(z_j, s) \right].$$

The next task is to get rid of $\int_0^t \int_{\mathbb{R}_0} z_j^2 d\eta_j(z_j, s)$ and take ∇_{x_k} out of the mathematical expectation in the formula above. To this end, for any $\lambda > 0$ let

$$\begin{aligned} Z^\lambda(t) &= \exp \left(- \sum_{j=1}^n \int_0^t \int_{\mathbb{R}_0} \lambda z_j^2 d\eta_j(z_j, s) - \sum_{j=1}^n \int_0^t \int_{\mathbb{R}_0} (e^{-\lambda z_j^2} - 1) d\nu_j(z_j) ds \right) \\ &= \exp \left(-\mathcal{A}(t) - \sum_{j=1}^n \int_0^t \int_{\mathbb{R}_0} (e^{-\lambda z_j^2} - 1) d\nu_j(z_j) ds \right). \end{aligned}$$

Thanks to [32, Theorem 1.4 and Remark 1.2] the process $[0, \infty) \ni t \mapsto Z^\lambda(t)$, is a martingale and one can define a new probability \mathbb{P}^λ on (Ω, \mathcal{F}) such that

$$\frac{d\mathbb{P}^\lambda}{d\mathbb{P}} \Big|_{\mathcal{F}_t} = Z^\lambda(t).$$

Moreover, under \mathbb{P}^λ the random measure η_j , $j \in \{1, \dots, n\}$, is a Poisson random measure with Lévy measure

$$\nu_j^\lambda(dz_j) = e^{-\lambda z_j^2} g_j(z_j) dz_j = e^{-\lambda z_j^2} \nu_j(dz_j),$$

and $d\tilde{\eta}_j^\lambda := d\eta_j - d\nu_j^\lambda$, where $d\nu_j^\lambda(z_j)dt = \nu_j^\lambda(dz_j)dt$, is a martingale measure. The solution under \mathbb{P} to the system (38) has, under \mathbb{P}^λ , the same law as the solution $\mathbf{X} := (\mathbf{X}^{(1)}, \dots, \mathbf{X}^{(n)})$ of the following system

$$(50) \quad \begin{aligned} d\mathbf{X}^{(i)}(t, x) &= \alpha^{(i)}(X(t, x))dt + \sum_{j=1}^n \int_{\mathbb{R}_0} \gamma_{ij}(z) (e^{-\lambda z_j^2} - 1) d\nu_j(z_j)dt + \sum_{j=1}^n \int_{\mathbb{R}_0} \gamma_{ij}(z) d\tilde{\eta}_j^\lambda(z_j, t), \\ \mathbf{X}^{(i)}(0) &= x_i, \quad i = 1, \dots, n. \end{aligned}$$

Arguing as in the proof of (49) we can show that

$$\nabla_{x_k} \mathbb{E}^\lambda \left[\Phi(\mathbf{X}(t, x)) \times \sum_{j=1}^n \int_0^t \int_{\mathbb{R}_0} z_j^2 d\eta_j(z_j, s) \right] = - \mathbb{E}^\lambda [\Phi(\mathbf{X}(t, x) J_\lambda^{(k)}(t))],$$

where

$$J_\lambda^{(k)}(t) := \sum_{j=1}^n \int_0^t \int_{\mathbb{R}_0} \frac{1}{e^{-\lambda z_j^2} g_j(z_j)} \frac{\partial \Lambda_{kj}^\lambda(s, z)}{\partial z_j} d\tilde{\eta}_j^\lambda(z_j, s),$$

and

$$\Lambda_{kj}^\lambda = e^{-\lambda z_j^2} z_j^2 \beta_j^{-1} \frac{\partial \mathbf{X}^{(j)}(s, x)}{\partial x_k}.$$

For the sake of simplicity, let us set

$$N_\lambda(t) = \sum_{j=1}^n \int_0^t \int_{\mathbb{R}_0} (e^{-\lambda z_j^2} - 1) d\nu_j(z_j) ds.$$

Note that

$$\begin{aligned} \int_0^\infty e^{N_\lambda(t)} \mathbb{E}^\lambda \left[\Phi(\mathbf{X}(t, x)) J_\lambda^{(k)}(t) \right] d\lambda &= \int_0^\infty e^{N_\lambda(t)} \mathbb{E} \left[Z^\lambda(t) \Phi(\mathbf{X}(t, x)) J_\lambda^{(k)}(t) \right] d\lambda \\ &= \int_0^\infty \mathbb{E} \left[e^{-\lambda \mathcal{A}(t)} \Phi(\mathbf{X}(t, x)) J_\lambda^{(k)}(t) \right] d\lambda. \end{aligned}$$

Since $d\tilde{\eta}_j^\lambda(z_j, s) = d\tilde{\eta}_j(z_j, s) + (1 - e^{\lambda z_j^2}) \nu_j(z_j) ds$, we can use integration-by-parts and Assumption 2.3-(2) to show that $J_\lambda^{(k)}(t) = J^{(k)}(t) - \lambda K^{(k)}(t)$. Hence

$$\int_0^\infty e^{N_\lambda(t)} \mathbb{E}^\lambda \left[\Phi(\mathbf{X}(t, x)) J_\lambda^{(k)}(t) \right] d\lambda = \int_0^\infty \mathbb{E} \left[e^{-\lambda \mathcal{A}(t)} \Phi(\mathbf{X}(t, x)) (J^{(k)}(t) - \lambda K^{(k)}(t)) \right] d\lambda.$$

Thanks to Fubini's theorem we infer that

$$\begin{aligned} \int_0^\infty e^{N_\lambda(t)} \mathbb{E}^\lambda \left[\Phi(\mathbf{X}(t, x)) J_\lambda^{(k)}(t) \right] d\lambda &= \mathbb{E} \left[\int_0^\infty e^{-\lambda \mathcal{A}(t)} \Phi(\mathbf{X}(t, x)) (J^{(k)}(t) - \lambda K^{(k)}(t)) d\lambda \right] \\ &= \mathbb{E} \left[\Phi(\mathbf{X}(t, x)) \frac{J^{(k)}(t)}{\mathcal{A}(t)} \right] - \mathbb{E} \left[\Phi(\mathbf{X}(t, x)) \frac{K^{(k)}(t)}{[\mathcal{A}(t)]^2} \right]. \end{aligned}$$

But, as in [45, Proof of Theorem 1] the following identity holds

$$\nabla_{x_k} \mathbb{E}[\Phi(\mathbf{X}(t, x))] = \nabla_{x_k} \mathbb{E} \left[\Phi(\mathbf{X}(t, x)) \frac{\mathcal{A}(t)}{\mathcal{A}(t)} \right] = \int_0^\infty \nabla_{x_k} \mathbb{E}[\Phi(\mathbf{X}(t, x)) \mathcal{A}(t) Z^\lambda(t)] e^{N_\lambda(t)} d\lambda,$$

from which we infer that

$$\begin{aligned} \nabla_{x_k} \mathbb{E}[\Phi(\mathbf{X}(t, x))] &= \int_0^\infty \nabla_{x_k} \mathbb{E}^\lambda [\Phi(\mathbf{X}(t, x)) \mathcal{A}(t)] e^{N_\lambda(t)} d\lambda \\ &= - \int_0^\infty e^{N_\lambda(t)} \mathbb{E}^\lambda [\Phi(\mathbf{X}(t, x)) J_\lambda^{(k)}(t)] d\lambda. \end{aligned}$$

Therefore,

$$\nabla_{x_k} \mathbb{E}[\Phi(\mathbf{X}(t, x))] = \mathbb{E} \left[\Phi(\mathbf{X}(t, x)) \frac{K^{(k)}(t)}{[\mathcal{A}(t)]^2} \right] - \mathbb{E} \left[\Phi(\mathbf{X}(t, x)) \frac{J^{(k)}(t)}{\mathcal{A}(t)} \right],$$

for any $k \in \{1, \dots, n\}$. This completes the proof of our lemma. \square

APPENDIX B. ESTIMATES OF $\nabla_x \mathbb{E}[\Phi(\mathbf{X}(t, x))]$

In this section we will derive estimates for the gradient of the Markov semigroup $\mathbb{E}[\Phi(\mathbf{X}(t, x))]$. Let $\{\lambda_j; j = 1, 2, \dots\}$ be a sequence of positive numbers, $\delta \in [0, \frac{1}{2}]$ and \mathbf{A}^δ be the matrix defined by

$$\mathbf{A}_{jk}^\delta = \begin{cases} \lambda_j^\delta & \text{if } j = k \\ 0 & \text{otherwise.} \end{cases}$$

Lemma B.1. *Assume that Assumption 2.3 and Assumption A.2 are verified instead. Then, for any $t > 0$ there exists a constant $C = C(t)$ such that*

$$(51) \quad |\nabla_x \mathbb{E}[\Phi(\mathbf{X}(t, x))]| \leq C(t) \left(\sum_{j=1}^n \beta_j^{-2} \lambda_j^{-2\delta} \right)^{\frac{1}{2}} |\Phi|_{\infty} \left[\mathbb{E} \int_0^t |A^{\delta} \nabla_x \mathbf{X}(s, x)|^2 ds \right]^{\frac{1}{2}},$$

for any $x \in \mathbb{R}^n$ and $\Phi \in C_b^2(\mathbb{R}^n)$.

Proof. For $p = 1, 2$ and $\delta \in [0, \frac{1}{2}]$ let

$$C_p(t) = \mathbb{E} \left(\frac{1}{[\mathcal{A}(t)]^{2p}} \right),$$

where

$$\mathcal{A}(t) = \sum_{j=1}^n \int_0^t \int_{\mathbb{R}_0} z_j^2 d\eta_j(z_j, s),$$

and

$$C_n = \sum_{j=1}^n \beta_j^{-2} \lambda_j^{-2\delta},$$

Before we proceed to the proof we should note that owing to Assumption 2.3 and Assumption A.2 we have that

$$(52) \quad \mathbb{E} \left[\sup_{s \in [0, t]} (|\mathbf{X}(s, x)|^2 + |A^{\delta} \nabla_x \mathbf{X}(s, x)|^2) \right] < \infty.$$

From the identity in Lemma A.3 we drive that

$$(53) \quad |\nabla_x \mathbb{E}[\Phi(\mathbf{X}(t, x))]|^2 \leq |\Phi|_{\infty}^2 \left(C_2(t) \sum_{k=1}^n \mathbb{E}[K^{(k)}(t)]^2 + C_1(t) \sum_{k=1}^n \mathbb{E}[J^{(k)}(t)]^2 \right).$$

Let us first estimate the term $\sum_{k=1}^n \mathbb{E}[J^{(k)}(t)]^2$. From discrete Cauchy-Schwarz inequality and Itô's isometry we derive that

$$\begin{aligned} \mathbb{E}[J^{(k)}(t)]^2 &\leq C \sum_{j=1}^n \beta_j^{-2} \lambda_j^{-2\delta} \sum_{j=1}^n \mathbb{E} \left(\int_0^t \int_{\mathbb{R}_0} \lambda_j^{\delta} \frac{1}{g_j(z_j)} \frac{\partial}{\partial z_j} [z_j^2 g_j(z_j)] \frac{\partial \mathbf{X}^{(j)}(s)}{\partial x_k} d\tilde{\eta}_j(z_j, s) \right)^2 \\ &\leq CC_n \sum_{j=1}^n \mathbb{E} \int_0^t \int_{\mathbb{R}_0} \left[\frac{1}{g_j(z_j)} \frac{\partial}{\partial z_j} (z_j^2 g_j(z_j)) \right]^2 \lambda_j^{2\delta} \left| \frac{\partial \mathbf{X}^{(j)}(s)}{\partial x_k} \right|^2 \nu(dz_j) ds \\ &\leq CC_n \int_{\mathbb{R}_0} \left[\frac{1}{g(z)} \frac{d}{dz} (z^2 g(z)) \right]^2 \nu(dz) \sum_{j=1}^n \int_0^t \lambda_j^{2\delta} \left| \frac{\partial \mathbf{X}^{(j)}(s)}{\partial x_k} \right|^2 ds. \end{aligned}$$

Since, by Assumption 2.3-(ii),

$$\begin{aligned} \left(\frac{1}{g(z)} \frac{d}{dz} [g(z) z^2] \right)^2 &= \left(\frac{g'(z)}{g(z)} z^2 + 2z \right)^2 \\ &\leq 2z^4 \left| \frac{g'(z)}{g(z)} \right|^2 + 4z^2 \\ &\leq c|z|^2 + c|z|^4, \end{aligned}$$

and $\int_{\mathbb{R}_0} |z|^p \nu(dz) < \infty$ for any $p \geq 2$ we infer that there exists $C > 0$ such that

$$(54) \quad \sum_{k=1}^n \mathbb{E}[J^{(k)}(t)]^2 \leq CC_n \sum_{j,k=1}^n \mathbb{E} \int_0^t \lambda_j^{2\delta} \left| \frac{\partial \mathbf{X}^{(j)}(s)}{\partial x_k} \right|^2 ds.$$

Now, we treat the term involving $K^k(t)$. Owing to Assumption 2.3-(iii) and (52) we can write

$$\begin{aligned} K^{(k)}(t) &= -2 \sum_{j=1}^n \int_0^t \int_{\mathbb{R}_0} z_j \beta_j^{-1} \frac{\partial \mathbf{X}^{(j)}(s)}{\partial x_k} d\tilde{\eta}_j(z_j, s) - 2 \sum_{j=1}^n \int_0^t \int_{\mathbb{R}_0} z_j \beta_j^{-1} \frac{\partial \mathbf{X}^{(j)}(s)}{\partial x_k} \nu_j(dz_j) ds \\ &= -2 \sum_{j=1}^n \int_0^t \int_{\mathbb{R}_0} z_j \beta_j^{-1} \frac{\partial \mathbf{X}^{(j)}(s)}{\partial x_k} d\tilde{\eta}_j(z_j, s) - 2 \int_{\mathbb{R}_0} z \nu(dz) \sum_{j=1}^n \int_0^t \beta_j^{-1} \frac{\partial \mathbf{X}^{(j)}(s)}{\partial x_k} ds \\ &=: K_1^{(k)}(t) + K_2^{(k)}(t). \end{aligned}$$

Hence

$$\mathbb{E}[K^{(k)}(t)]^2 \leq 2\mathbb{E}[K_1^{(k)}(t)]^2 + 2\mathbb{E}[K_2^{(k)}(t)]^2.$$

Let us first deal with $\mathbb{E}[K_1^{(k)}(t)]^2$. Using the discrete Cauchy-Schwarz inequality and Itô's isometry we obtain

$$\begin{aligned} \mathbb{E}[K_1^{(k)}(t)]^2 &\leq C \sum_{j=1}^n \beta_j^{-2} \lambda_j^{-2\delta} \times \sum_{j=1}^n \mathbb{E} \left(\int_0^t \int_{\mathbb{R}_0} z_j \lambda_j^{2\delta} \frac{\partial \mathbf{X}^{(j)}(s)}{\partial x_k} d\tilde{\eta}_j(z_j, s) \right)^2 \\ &\leq CC_n \sum_{j=1}^n \mathbb{E} \left[\int_0^t \int_{\mathbb{R}_0} z_j^2 \lambda_j^{2\delta} \left| \frac{\partial \mathbf{X}^{(j)}(s)}{\partial x_k} \right|^2 \nu_j(dz_j) ds \right] \\ &\leq CC_n \int_{\mathbb{R}_0} z^2 \nu(dz) \sum_{j=1}^n \mathbb{E} \left[\int_0^t \lambda_j^{2\delta} \left| \frac{\partial \mathbf{X}^{(j)}(s)}{\partial x_k} \right|^2 ds \right]. \end{aligned}$$

Owing to the discrete Cauchy-Schwarz inequality we have

$$\begin{aligned} \mathbb{E}[K_2^{(k)}(t)]^2 &\leq \left(\int_{\mathbb{R}_0} z \nu(dz) \right)^2 \sum_{j=1}^n \beta_j^{-2} \lambda_j^{-2\delta} \times \sum_{j=1}^n \mathbb{E} \left(\int_0^t \lambda_j^\delta \frac{\partial \mathbf{X}^{(j)}(s)}{\partial x_k} ds \right)^2 \\ &\leq \left(\int_{\mathbb{R}_0} z \nu(dz) \right)^2 \sum_{j=1}^n \beta_j^{-2} \lambda_j^{-2\delta} \times t \sum_{j=1}^n \mathbb{E} \left[\int_0^t \lambda_j^{2\delta} \left| \frac{\partial \mathbf{X}^{(j)}(s)}{\partial x_k} \right|^2 ds \right]. \end{aligned}$$

Hence there exists $C > 0$ such that

$$(55) \quad \sum_{k=1}^n \mathbb{E}[K^{(k)}(t)]^2 \leq CC_n(1+t) \sum_{j,k=1}^n \mathbb{E} \left[\int_0^t \lambda_j^{2\delta} \left| \frac{\partial \mathbf{X}^{(j)}(s)}{\partial x_k} \right|^2 ds \right].$$

One can derive from (55) and (54) that

$$\begin{aligned} \sum_{k=1}^n \mathbb{E}[K^{(k)}(t)]^2 &\leq CC_n(1+t) \mathbb{E} \int_0^t |\mathbf{A}^\delta \nabla_x \mathbf{X}(s)|^2 ds, \\ \sum_{k=1}^n \mathbb{E}[J^{(k)}(t)]^2 &\leq CC_n \mathbb{E} \int_0^t |\mathbf{A}^\delta \nabla_x \mathbf{X}(s)|^2 ds. \end{aligned}$$

Hence, by plugging these last identities in (53) we deduce that there exists C such that

$$(56) \quad |\nabla_x \mathbb{E}[\Phi(\mathbf{X}(t, x))]| \leq CC_n^{\frac{1}{2}} |\Phi|_\infty \left[\mathbb{E} \int_0^t |\mathbf{A}^\delta \nabla_x \mathbf{X}(s, x)|^2 ds \right]^{\frac{1}{2}} (C_2^{\frac{1}{2}}(t)(1+t)^{\frac{1}{2}} + C_1^{\frac{1}{2}}(t))$$

Now it remains to prove that $C_p(t) = \mathbb{E} \mathcal{A}(t)^{-2p}$ is finite. Since the measures η_j , $j = 1, 2, \dots$ are positive on \mathbb{R}_0 and $z_j^2 > 0$ we easily see that

$$\mathcal{A}(t) \geq \int_0^t \int_{\mathbb{R}_0} z_1^2 d\eta_1(z_1, s).$$

Thus, by Assumption 2.3-(iv) and [45, Remark 3.2] it follows that

$$\mathbb{E} \mathcal{A}(t)^{-2p} \leq C \left(t^{-2p} + t^{-\frac{4p}{\alpha}} \right),$$

and therefore,

$$C_p(t) \leq C \left(t^{-2p} + t^{-\frac{4p}{\alpha}} \right) < \infty.$$

This ends the proof of our lemma. \square

APPENDIX C. PROOF THAT \mathbf{u}_n^R CONVERGES TO \mathbf{u}^R STRONGLY IN $L^2(0, T; \mathbf{H})$

In this section we are aiming to prove that the Galerkin solution \mathbf{u}_n^R to (29) converges to the solution \mathbf{u}^R of (23). To do so we consider the following system of finite dimensional differential equations

$$(57) \quad \begin{aligned} \frac{d}{dt} \mathbf{v}_n^R(t) + A \mathbf{v}_n^R(t) + \rho(|\mathbf{v}_n^R(t) + \Pi_n \mathfrak{S}(t)|^2 / R) \Pi_n \mathbf{B}(\mathbf{v}_n^R(t) + \Pi_n \mathfrak{S}(t), \mathbf{v}_n^R(t) + \Pi_n \mathfrak{S}(t)) &= 0, \\ \mathbf{v}_n^R(0) &= \Pi_n \xi \in \mathbf{H}_n, \end{aligned}$$

where $\mathfrak{S} \in L^\infty(0, T; \mathbf{H})$.

Lemma C.1. *For any $\xi \in \mathbf{H}$, $\mathfrak{S} \in L^\infty(0, T; \mathbf{H})$ there exists $C > 0$ such that*

$$\sup_{n \in \mathbb{N}, R > 0} [\|\mathbf{v}_n^R\|_{C(0, T; \mathbf{H})} + \|\mathbf{v}_n^R\|_{L^2(0, T; \mathbf{V})}] < C,$$

Proof. Multiplying (57) by \mathbf{v}_n^R yields that

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} |\mathbf{v}_n^R(t)|^2 + \kappa |A^{\frac{1}{2}} \mathbf{v}_n^R(t)|^2 \\ = - \langle \rho(|\mathbf{v}_n^R(t) + \Pi_n \mathfrak{S}(t)|^2 / R) \Pi_n \mathbf{B}(\mathbf{v}_n^R(t) + \Pi_n \mathfrak{S}(t), \mathbf{v}_n^R(t) + \Pi_n \mathfrak{S}(t)), \mathbf{v}_n^R(t) \rangle. \end{aligned}$$

Since ρ is bounded by 1 and $\mathbf{B}(\cdot, \cdot)$ is bilinear we infer that

$$\begin{aligned} | - \langle \rho(|\mathbf{v}_n^R(t) + \Pi_n \mathfrak{S}(t)|^2 / R) \Pi_n \mathbf{B}(\mathbf{v}_n^R(t) + \Pi_n \mathfrak{S}(t), \mathbf{v}_n^R(t) + \Pi_n \mathfrak{S}(t)), \mathbf{v}_n^R(t) \rangle | \\ \leq | \langle \Pi_n \mathbf{B}(\mathbf{v}_n^R(t) + \Pi_n \mathfrak{S}(t), \mathbf{v}_n^R(t) + \Pi_n \mathfrak{S}(t)), \mathbf{v}_n^R(t) \rangle | \\ \leq | \langle \Pi_n \mathbf{B}(\mathbf{v}_n^R(t) + \Pi_n \mathfrak{S}(t), \mathbf{v}_n^R(t) + \Pi_n \mathfrak{S}(t)), \mathbf{v}_n^R(t) \rangle | \\ + | \langle \Pi_n \mathbf{B}(\mathbf{v}_n^R(t), \Pi_n \mathfrak{S}(t)), \mathbf{v}_n^R(t) \rangle | \\ + | \langle \Pi_n \mathbf{B}(\Pi_n \mathfrak{S}(t), \Pi_n \mathfrak{S}(t)), \mathbf{v}_n^R(t) \rangle |. \end{aligned}$$

Now using Assumption 2.2-(a), Assumption 2.2-(c) and the fact $\|\Pi_n\|_{\mathcal{L}(\mathbf{V}^*, \mathbf{V}^*)} \leq 1$ we derive that

$$\begin{aligned} | - \langle \rho(|\mathbf{v}_n^R(t) + \Pi_n \mathfrak{S}(t)|^2 / R) \Pi_n \mathbf{B}(\mathbf{v}_n^R(t) + \Pi_n \mathfrak{S}(t), \mathbf{v}_n^R(t) + \Pi_n \mathfrak{S}(t)), \mathbf{v}_n^R(t) \rangle | \\ \leq C |\mathbf{v}_n^R(t)| |\mathfrak{S}(t)| |A^{\frac{1}{2}} \mathbf{v}_n^R(t)| + C |\mathfrak{S}(t)|^2 |A^{\frac{1}{2}} \mathbf{v}_n^R(t)|. \end{aligned}$$

Hence, by the Young inequality we deduce that for any $\varepsilon > 0$ there exists a constant $C_\varepsilon > 0$ such that

$$\begin{aligned} | - \langle \rho(|\mathbf{v}_n^R(t) + \Pi_n \mathfrak{S}(t)|^2 / R) \Pi_n \mathbf{B}(\mathbf{v}_n^R(t) + \Pi_n \mathfrak{S}(t), \mathbf{v}_n^R(t) + \Pi_n \mathfrak{S}(t)), \mathbf{v}_n^R(t) \rangle | \\ \leq C_\varepsilon |\mathfrak{S}(t)|^2 |\mathbf{v}_n^R(t)|^2 + C_\varepsilon |\mathfrak{S}(t)|^4 + \varepsilon |A^{\frac{1}{2}} \mathbf{v}_n^R(t)|^2. \end{aligned}$$

Therefore, by choosing $\varepsilon = \kappa$ we infer that there exists a number $C_\kappa > 0$ such that

$$\frac{d}{dt} |\mathbf{v}_n^R(t)|^2 + \kappa |A^{\frac{1}{2}} \mathbf{v}_n^R(t)|^2 \leq C_\kappa |\mathbf{v}_n^R(t)|^2 |\mathfrak{S}(t)|^2 + C_\kappa |\mathfrak{S}(t)|^4.$$

Owing to this last inequality and the Gronwall's inequality we obtain

$$\begin{aligned} \sup_{s \in [0, T]} |\mathbf{v}_n^R(s)|^2 &\leq (|\xi|^2 + C_\kappa \sup_{s \in [0, t]} |\mathfrak{S}(s)|^4) e^{\sup_{s \in [0, T]} |\mathfrak{S}(s)|^2 T} =: C^*, \\ \int_0^T |A^{\frac{1}{2}} \mathbf{v}_n^R(s)|^2 ds &\leq |\xi|^2 + C_\kappa \sup_{s \in [0, T]} |\mathfrak{S}(s)|^2 T (C^* + \sup_{s \in [0, T]} |\mathfrak{S}(s)|^2), \end{aligned}$$

which implies the estimate in Lemma C.1. \square

Thanks to Lemma C.1 it is not difficult to prove that there exists $C > 0$ such that

$$\sup_{n \in \mathbb{N}, R > 0} \left\| \frac{d\mathbf{v}_n^R}{dt} \right\|_{\mathbf{V}^*} \leq C.$$

This estimate, the one in Lemma C.1 and Aubin-Lions lemma imply that there exists $\mathbf{v} \in L^2(0, T; \mathbf{H})$ and one can extract subsequence, still denote by \mathbf{v}_n^R , such that

$$(58) \quad \mathbf{v}_n^R \rightarrow \mathbf{v} \text{ strongly in } L^2(0, T; \mathbf{H}).$$

Also

$$(59) \quad \mathbf{v}_n^R \rightarrow \mathbf{v} \text{ weakly-star in } L^\infty(0, T; \mathbf{H}),$$

$$(60) \quad \mathbf{v}_n^R \rightarrow \mathbf{v} \text{ weakly in } L^2(0, T; \mathbf{V}).$$

Now we can easily show that \mathbf{v} solves (31). Now let \mathfrak{S} be the stochastic convolution defined in (20)-(21). The stochastic process $\mathbf{u}_n^R = \mathbf{v}_n^R + \Pi_n \mathfrak{S}$ solves (29) and thanks to (58) we see that

$$(61) \quad \mathbf{u}_n^R \rightarrow \mathbf{u}^R \text{ strongly in } L^2(0, T; \mathbf{H}) \text{ } \mathbb{P} - a.s.,$$

where \mathbf{u}^R is the unique solution to (23).

APPENDIX D. ACKNOWLEDGMENT

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